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Heat Determinant on Manifolds

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We introduce and study new invariants associated with Laplace type elliptic partial differential operators on manifolds. These invariants are constructed by using the off-diagonal heat kernel; they are not pure spectral invariants, that is, they depend not only on the eigenvalues but also on the corresponding eigenfunctions in a non-trivial way. We compute the first three low-order invariants explicitly.

1 Introduction

The heat kernel is one of the most important tools of global analysis, spectral geometry, differential geometry and mathematical physics, in particular, quantum field theory [16, 8, 14, 17]. In quantum field theory the main objects of interest are described by the Green functions of self-adjoint elliptic partial differential operators on manifolds and their spectral invariants such as the functional determinants. In spectral geometry one is interested in the relation of the spectrum of natural elliptic partial differential operators to the geometry of the manifold, more precisely, one studies the question: “To what extent does the spectrum of a differential operator determine the geometry of the underlying manifold?”

There are also non-trivial links between the spectral invariants and the non-linear completely integrable evolution systems, such as Korteweg-de Vries hierarchy (see, e.g. [17]). In many interesting cases such systems are, in fact, infinite-dimensional Hamiltonian systems, and the spectral invariants of a linear elliptic partial differential operator are nothing but the integrals of motion of this system.

Instead of studying the spectrum of a differential operator directly one usually studies its spectral functions, that is, spectral traces of some functions of the operator, such as the zeta function, and the heat trace. Usually one does not know the spectrum exactly; that is why, it becomes very important to study various asymptotic regimes. It is well known, for example, that one can get information about the asymptotic properties of the spectrum by studying the short time asymptotic expansion of the heat trace. The coefficients of this expansion, called the heat trace coefficients (or global heat kernel coefficients), play very important role in spectral geometry and mathematical physics [17, 14].

The simplest case of a Laplace operator on a compact manifold without boundary is well understood and there is a vast literature on this subject, see [14] and the references therein. For a Laplace type operator on a compact manifold without boundary there is a well defined local asymptotic expansion of the heat kernel, which enables one to compute its diagonal and then the heat trace by directly integrating the heat kernel diagonal; this gives all heat trace coefficients. However, many ideas and techniques do not apply directly in more general cases.

The existence of non-isometric isospectral manifolds demonstrates that the *spectrum alone does not determine the geometry* (see, e.g. [7]). That is why, we propose to study more general invariants of partial differential operators that are not spectral invariants, that is, they depend not only on the eigenvalues but also on the eigenfunctions, and, therefore, contain much more information about the geometry of the manifold.

In this paper we propose to study *new heat invariants* of second-order Laplace type elliptic partial differential operators acting on sections of vector bundles over Riemannian manifolds. Our goal is to develop a comprehensive methodology for such invariants in the same way as the theory of the standard heat trace invariants. Namely, we will define and study new heat invariants of differential operators and compute explicitly some leading terms of the asymptotic expansion of new heat invariants.

Our main result can be formulated as follows.

Theorem 1 *Let (M, g) be a smooth compact n -dimensional Riemannian manifold without boundary with metric g and \mathcal{V} be a vector bundle over M of dimension N . Let ∇ be a connection on the vector bundle and Q be an endomorphism of the bundle \mathcal{V} . Let $\Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu$ be the Laplace operator and $L : C^\infty(\mathcal{V}) \rightarrow C^\infty(\mathcal{V})$ be the Laplace type partial differential operator of the form $L = -\Delta + Q$. Let $U(t; x, x')$ be the heat kernel of the operator L ,*

$$P_{\mu\nu'}(t; x, x') = \text{tr } U^*(t; x, x')\nabla_\mu\nabla_{\nu'}U(t; x, x'), \quad (1.1)$$

where tr is the fiber trace, and $K(t)$ be the functional defined by

$$K(t) = \int_{M \times M} dx dx' \det P_{\mu\nu'}(t; x, x'). \quad (1.2)$$

Then there is an asymptotic expansion as $t \rightarrow 0$

$$K(t) \sim (4\pi)^{-n^2} \left(\frac{\pi}{2n}\right)^{n/2} t^{-n(n+\frac{1}{2})} \sum_{k=0}^{\infty} t^k B_k, \quad (1.3)$$

where

$$B_k = \int_M dv b_k, \quad (1.4)$$

dv is the Riemannian volume element and b_k are differential polynomials in the Riemann curvature, the bundle curvature and the endomorphism Q with some universal numerical coefficients that depend only on the dimensions n and N . The low order coefficients are

$$b_0 = \frac{1}{2}N^n, \quad (1.5)$$

$$b_1 = N^n \frac{12n^2 - n + 10}{72n} R - nN^{n-1} \text{tr } Q, \quad (1.6)$$

$$\begin{aligned}
b_2 = & N^n \left\{ \frac{20n^4 - 8n^3 - 11n^2 - 6n + 6}{144n^2} R^2 + \frac{4n^3 + 11n^2 + n - 4}{120n^2} \nabla_\mu \nabla^\mu R \right. \\
& + \frac{-24n^3 + 84n^2 - 576n + 385}{4320n^2} R_{\mu\nu} R^{\mu\nu} + \frac{8n^3 - 8n^2 - 18n + 15}{1440n^2} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \Big\} \\
& + \frac{n^3 - n^2 + 3n - 12}{12n^2} N^{n-1} \text{tr} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + \frac{-12n^3 - 4n^2 + n + 2}{12n} N^{n-1} R \text{tr} Q \\
& - \frac{n+2}{6} N^{n-1} \text{tr} \nabla^\mu \nabla_\mu Q + n N^{n-1} \text{tr} Q^2 + n(n-1) N^{n-2} (\text{tr} Q)^2. \tag{1.7}
\end{aligned}$$

Here $R_{\mu\nu\alpha\beta}$, $R_{\mu\nu}$ and R are the Riemann tensor, the Ricci tensor and the scalar curvature respectively, and $\mathcal{R}_{\mu\nu}$ is the curvature of the bundle connection.

Of course, the derivative terms can be neglected on manifolds without boundary.

This paper is organized as follows. In Sec. 2 we describe the necessary geometric framework and define the heat kernel and some invariants, such as the heat trace and the heat content. In Sec. 3. we define a new invariant called the heat determinant for scalar operators and show that on manifolds without boundary it is trivial. In Sec. 4 we define an alternative invariant that we call the heat determinant on vector bundles. In Sec. 5 we introduce the machinery of standard off-diagonal heat kernel asymptotics and compute the heat content asymptotics. In Sec. 6 we compute the determinant of the mixed derivatives of the heat kernel. In Sec. 7 we establish the asymptotics of the heat determinant and in Sec. 8 we compute some low-order coefficients of this expansion. Some of the technical formulas for the derivatives of the Synge function and the parallel transport operator are listed in the Appendix.

2 Heat Kernel

Let (M, g) be a compact Riemannian manifold of dimension n and \mathcal{V} be a vector bundle over the manifold M with a typical fiber V of dimension N . First of all, we fix notation. We denote the local coordinates in a chart by $x = (x^\mu)$, $\mu = 1, \dots, n$, and use the Einstein summation convention. The components of the metric in the coordinate basis are denoted by $g_{\mu\nu}$, the determinant of the metric is denoted by $g = \det g_{\mu\nu}$, the Levi-Civita symbol is denoted by $\varepsilon_{\mu_1 \dots \mu_n}$, the covariant Levi-Civita symbols are denoted by $E_{\mu_1 \dots \mu_n} = g^{1/2} \varepsilon_{\mu_1 \dots \mu_n}$ and $E^{\mu_1 \dots \mu_n} = g^{-1/2} \varepsilon^{\mu_1 \dots \mu_n}$. Let $dx = dx^1 \cdots dx^n$ be the Lebesgue measure in a local chart, and $dV = dx g^{1/2}$, be the Riemannian volume element; further, let $\langle \varphi, \psi \rangle = \text{tr} \varphi^* \otimes \psi$ be the fiber inner

product, where φ^* is the dual section and tr is the fiber trace. We use parenthesis to denote symmetrization over all indices included and square brackets to denote the anti-symmetrization. The indices excluded from (anti)-symmetrization are separated by vertical lines.

We will consider two-point functions and tensors such as $G(x, x')$. Then the derivatives of such two functions with respect to x^μ will be denoted by $\nabla_\mu G$ (or by a semicolon $G_{;\mu}$) and the derivatives with respect to x'^ν will be denoted by $\nabla_{\nu'} G$ (or by a semicolon $G_{;\nu'}$). For convenience we introduce special notation for the coincidence limit of two-point functions; we will denote it by square brackets

$$[f] = \lim_{x \rightarrow x'} f(x, x'). \quad (2.1)$$

In case when there is a boundary ∂M , we assume that the boundary is smooth so that the inward pointing unit normal N is well defined. We use the natural orientation of the boundary and denote the local coordinates on the boundary by $\hat{x} = (\hat{x}^i)$, $i = 1, \dots, n-1$, so that the local equation of the boundary is $x^\mu = x^\mu(\hat{x})$. The Lebesgue measure on the boundary is denoted by $d\hat{x} = d\hat{x}^1 \dots d\hat{x}^{n-1}$. The induced Riemannian metric on the boundary is denoted by $\hat{g}_{ij} = \frac{\partial x^\mu}{\partial \hat{x}^i} \frac{\partial x^\nu}{\partial \hat{x}^j} g_{\mu\nu}$ and the Riemannian volume element on the boundary by $d\hat{v} = d\hat{x} \hat{g}^{1/2}$, where $\hat{g} = \det \hat{g}_{ij}$.

Let $C^\infty(\mathcal{V})$ be the space of smooth sections of the bundle \mathcal{V} . We define a natural invariant L^2 inner product on $C^\infty(\mathcal{V})$ by $(\varphi, \psi) = \int_M d\nu \langle \varphi, \psi \rangle$ and the corresponding norm $\|\varphi\| = \sqrt{(\varphi, \varphi)}$. The completion of the space $C^\infty(\mathcal{V})$ in this norm defines the Hilbert space $L^2(\mathcal{V})$. The L^2 trace of a trace-class operator G will be denoted by Tr , that is,

$$\text{Tr } G = \int_M d\nu \text{tr } G(x, x), \quad (2.2)$$

where tr is the fiber trace and $G(x, x')$ is the integral kernel of the operator G .

The components of a connection one form on the vector bundle \mathcal{V} are denoted by \mathcal{A}_μ . We introduce the covariant exterior derivative of sections of \mathcal{V} by

$$D\varphi = d\varphi + \mathcal{A} \wedge \varphi. \quad (2.3)$$

Then, obviously,

$$D^2\varphi = \mathcal{R}\varphi, \quad (2.4)$$

where

$$\mathcal{R} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \quad (2.5)$$

is the curvature of the connection \mathcal{A} .

Let $L : C^\infty(\mathcal{V}) \rightarrow C^\infty(\mathcal{V})$ be a second-order formally self-adjoint elliptic partial differential operator with a positive definite leading symbol. If the leading symbol of the operator L is scalar then the operator L is called of Laplace type; in this case one can define a Riemannian metric $g_{\mu\nu}$ by the leading symbol of the operator L , a connection ∇ on the vector bundle \mathcal{V} and a self-adjoint endomorphism Q of the vector bundle \mathcal{V} so that

$$L = -\Delta + Q, \quad (2.6)$$

where $\Delta = g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the Laplacian. In the case when there is a (smooth) boundary ∂M we assume that some suitable boundary conditions are imposed, either Dirichlet

$$\varphi|_{\partial M} = 0 \quad (2.7)$$

or Neumann

$$\nabla_N \varphi|_{\partial M} = 0. \quad (2.8)$$

Here, $\nabla_N = N^\mu \nabla_\mu$ is the normal derivative and N^μ is the inward pointing unit normal to the boundary.

It is well known [14] that if the manifold M is compact then the operator L has only point spectrum consisting of real discrete eigenvalues with finite multiplicities bounded from below

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots, \quad (2.9)$$

where each eigenvalue is counted with its (finite) multiplicity. The spectrum $(\lambda_k, \varphi_k)_{k=1}^\infty$ of the operator L is determined by

$$L\varphi_k = \lambda_k \varphi_k; \quad (2.10)$$

without loss of generality we can take the sequence of the eigensections to be orthonormal, that is,

$$(\varphi_k, \varphi_m) = \delta_{km}. \quad (2.11)$$

Then for $t > 0$ the heat semigroup $\exp(-tL)$ is a bounded operator with the integral kernel

$$U(t; x, x') = \sum_{k=1}^{\infty} e^{-t\lambda_k} \varphi_k(x) \otimes \varphi_k^*(x'), \quad (2.12)$$

called the heat kernel of the operator L . The heat kernel satisfies the heat equation

$$(\partial_t + L)U(t; x, x') = 0 \quad (2.13)$$

with the initial condition

$$U(0; x, x') = \delta(x, x'), \quad (2.14)$$

where we use the notation $\delta(x, x') = g^{-1/4}(x)g^{-1/4}(x')\delta(x - x')$ for the covariant delta-function.

We would like to define some invariants of the operator L , that we call heat invariants, constructed entirely from the heat kernel of the operator L without any additional ingredients. There are two type of invariants. The usual ones, called spectral invariants, depend only on the eigenvalues and do not depend on the eigensections. More general heat invariants depend on both the eigenvalues and the eigensections.

One of the best known invariants is the *heat trace* which is obtained by integrating of the heat kernel diagonal

$$\Theta(t) = \text{Tr} \exp(-tL) = \int_M dv \text{tr} U(t; x, x) = \sum_{k=1}^{\infty} e^{-t\lambda_k}. \quad (2.15)$$

This is obviously a spectral invariant of the operator L since it only depends on the eigenvalues of the operator but does not depend on the eigenfunctions.

For a scalar operator acting on smooth real functions one can define another invariant (called the *heat content*) by integrating the off-diagonal heat

$$\Pi(t) = \int_{M \times M} dv dv' U(t; x, x') = \sum_{k=1}^{\infty} e^{-t\lambda_k} |\Phi_k|^2, \quad (2.16)$$

where

$$\Phi_k = \int_M dv \varphi_k. \quad (2.17)$$

For general operators acting on sections of a vector bundle this obviously does not work since the off-diagonal heat kernel is not a scalar function and the trace of the off-diagonal heat kernel is not invariant. One could, of course, define an invariant with the help of a section ψ of the bundle \mathcal{V} by

$$\tilde{\Pi}(t; \psi) = \int_{M \times M} dv dv' \langle \psi, U(t; x, x') \psi(x') \rangle, \quad (2.18)$$

with $\langle \cdot, \cdot \rangle$ the fiber inner product, however, this introduces an additional ingredient, namely, an additional section, which is not an invariant of the operator only. What we try to do in this paper is rather different, we want to define an invariant entirely in terms of the heat kernel without the need for any additional ingredients.

3 Heat Determinant of Scalar Operators

Let us first consider a general scalar Laplace type operator (2.6) acting on real smooth functions. We propose to study a new invariant defined as follows. First, we introduce the tensor of mixed derivatives of the heat kernel

$$\tilde{P}_{\mu\nu'}(t; x, x') = \nabla_\mu \nabla_{\nu'} U(t; x, x'). \quad (3.1)$$

Its spectral representation takes the form

$$\tilde{P}_{\mu\nu'}(t; x, x') = \sum_{k=1}^{\infty} e^{-t\lambda_k} \nabla_\mu \varphi_k(x) \nabla_{\nu'} \varphi_k(x'). \quad (3.2)$$

Now, we notice that $\det \tilde{P}_{\mu\nu'}$ is a bi-scalar density of weight 1. Therefore, we can define a new invariant (that we call the *heat determinant*) by integrating the determinant with the non-invariant Lebesgue measure dx instead of the invariant Riemannian measure $dv = dx g^{1/2}$

$$\tilde{K}(t) = \int_{M \times M} dx dx' \det \tilde{P}_{\mu\nu'}(t; x, x'). \quad (3.3)$$

Further, we define some new invariants that measure the correlations between eigenfunctions,

$$\tilde{\Phi}_{k_1 \dots k_n} = \int_M d\varphi_{k_1} \wedge \dots \wedge d\varphi_{k_n}. \quad (3.4)$$

By using the equation

$$d\varphi_{k_1} \wedge \dots \wedge d\varphi_{k_n} = d(\varphi_{k_1} d\varphi_{k_2} \wedge \dots \wedge d\varphi_{k_n}), \quad (3.5)$$

and the Stokes theorem it is easy to see that

$$\tilde{\Phi}_{k_1 \dots k_n} = \int_{\partial M} \varphi_{k_1} d\varphi_{k_2} \wedge \dots \wedge d\varphi_{k_n}. \quad (3.6)$$

Thus, for manifolds without boundary or with Dirichlet boundary conditions all invariants $\tilde{\Phi}_{k_1 \dots k_n}$ vanish,

$$\tilde{\Phi}_{k_1 \dots k_n} = 0. \quad (3.7)$$

Also, it is obvious that the invariants $\tilde{\Phi}_{k_1 \dots k_n}$ are completely antisymmetric in all their indices k_1, \dots, k_n and, therefore, vanish if any of the indices are equal.

Now, by using the definition of the heat determinant one can express the heat determinant in terms of the spectral form

$$\tilde{K}(t) = \frac{1}{n!} \sum_{k_1, \dots, k_n=1}^{\infty} \exp \{-t(\lambda_{k_1} + \dots + \lambda_{k_n})\} |\tilde{\Phi}_{k_1 \dots k_n}|^2, \quad (3.8)$$

Further, by using the antisymmetry of the invariants $\tilde{\Phi}_{k_1 \dots k_n}$ we can reorder the indices in the increasing order, which leads to a combinatorial factor $n!$; thus,

$$\tilde{K}(t) = \sum_{1 \leq k_1 < k_2 < \dots < k_n} \exp \{-t(\lambda_{k_1} + \dots + \lambda_{k_n})\} |\tilde{\Phi}_{k_1 \dots k_n}|^2. \quad (3.9)$$

One can show that for the matrix $\tilde{P} = (\tilde{P}_{\mu\nu'})$ of mixed derivatives $\tilde{P}_{\mu\nu'} = \nabla_\mu \nabla_{\nu'} U$ of any scalar two-point function U there holds

$$\det \tilde{P} = \frac{1}{n} \partial_\mu \partial_{\nu'} \left(U(\det \tilde{P}) \tilde{P}^{-1\nu'\mu} \right), \quad (3.10)$$

where $\tilde{P}^{-1\nu'\mu}$ is the inverse matrix. Indeed, we have

$$\det \tilde{P} = \frac{1}{n} \partial_\mu \partial_{\nu'} G^{\nu'\mu}, \quad (3.11)$$

where

$$G^{\nu'\mu} = \frac{1}{(n-1)!} \varepsilon^{\mu\mu_1 \dots \mu_{n-1}} \varepsilon^{\nu'\nu'_1 \dots \nu'_{n-1}} \tilde{P}_{\mu_1\nu'_1} \dots \tilde{P}_{\mu_{n-1}\nu'_{n-1}} U. \quad (3.12)$$

This tensor is defined for an arbitrary matrix \tilde{P} . One can easily show that in the case when the matrix \tilde{P} is not degenerate this tensor is equal to

$$G^{\nu'\mu} = (\det \tilde{P}) \tilde{P}^{-1\nu'\mu} U. \quad (3.13)$$

Therefore, by using the Stokes theorem, the heat determinant takes the form

$$\tilde{K}(t) = \int_{\partial M \times \partial M} d\hat{x} d\hat{x}' \frac{1}{n} N_\mu N_{\nu'} G^{\nu'\mu}(t; x, x'). \quad (3.14)$$

In particular, for manifolds without boundary and for Dirichlet boundary conditions the heat determinant vanishes

$$\tilde{K}(t) = 0. \quad (3.15)$$

4 Heat Determinants on Vector Bundles

Notice that the definition of the heat determinant above does not work for general operators acting on sections of a vector bundle. This is because the mixed derivative $\nabla_\mu \nabla_{\nu'} U(t; x, x')$ is a section of the external product bundle $\mathcal{V} \boxtimes \mathcal{V}^*$ (that is, it is a section of \mathcal{V} at the point x and a dual section at the point x' , in addition to being a covector at both these points). Therefore, we need to modify it accordingly. Also, we would like to have an invariant that does not vanish on manifolds without boundary. The heat determinant can now be defined as follows. First, to define an invariant tensor we multiply the mixed derivative of the heat kernel by the dual heat kernel

$$P_{\mu\nu'}(t; x, x') = \text{tr } U^*(t; x, x') \nabla_\mu \nabla_{\nu'} U(t; x, x'). \quad (4.1)$$

Recall that for a self-adjoint operator L

$$U^*(t; x, x') = U(t; x', x). \quad (4.2)$$

This quantity is the bi-covector at the points x and x' , and, therefore, its determinant is a bi-scalar density. Thus, in the same way as we defined the heat determinant for the scalar operators we can define

$$K(t) = \int_{M \times M} dx dx' \det P_{\mu\nu'}(t; x, x'). \quad (4.3)$$

Let us define the one-forms $\Phi_l^k = \Phi_{l\mu}^k dx^\mu$ by

$$\Phi_l^k = \langle \varphi_k, D\varphi_l \rangle, \quad (4.4)$$

where

$$\Phi_{l\mu}^k = \langle \varphi_k, \nabla_\mu \varphi_l \rangle, \quad (4.5)$$

and the invariants

$$C_{l_1 \dots l_n}^{k_1 \dots k_n} = \int_M \Phi_{l_1}^{k_1} \wedge \dots \wedge \Phi_{l_n}^{k_n}. \quad (4.6)$$

Then the tensor (4.1) takes the form

$$P_{\mu\nu'}(t; x, x') = \sum_{k,l=1}^{\infty} e^{-t(\lambda_k + \lambda_l)} \Phi_{l\mu}^k(x) \left(\Phi_{l\nu'}^k(x') \right)^*, \quad (4.7)$$

and the heat determinant can be written in the form

$$K(t) = \frac{1}{n!} \sum_{k_1, l_1, \dots, k_n, l_n=1}^{\infty} \exp \{-t(\lambda_{k_1} + \lambda_{l_1} + \dots + \lambda_{k_n} + \lambda_{l_n})\} |C_{l_1 \dots l_n}^{k_1 \dots k_n}|^2. \quad (4.8)$$

The advantage of this invariant is that it is defined entirely in terms of the heat kernel without the need for any additional ingredients, like a section or the parallel transport operator or another differential operator. Further, we can define the corresponding zeta function by the combined Laplace-Mellin transform, that is,

$$Z(s, \lambda) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} e^{t\lambda} K(t), \quad (4.9)$$

where λ has a sufficiently large negative real part and s has a sufficiently large positive real part. Also, this heat determinant does not vanish for manifolds without boundary and defines a new invariant of operators on manifolds without boundary.

5 Heat Kernel Asymptotics

We consider manifolds without boundary. We will extensively use the machinery of two-point geometric functions such as the Synge function (see, for example [20, 11, 2, 5]). The Synge function $\sigma(x, x')$ is defined as one-half the square of the geodesic distance between the points x and x' . At least for sufficiently close points x and x' this function is well defined and smooth.

We use the following notation: each additional index denotes the covariant derivative with respect to x^μ and each primed index denotes the covariant derivative with respect to x'^ν , e.g. $\sigma_\mu = \nabla_\mu \sigma$, $\sigma_{\nu'} = \nabla_{\nu'} \sigma$, $\sigma_{\mu\nu'} = \nabla_\mu \nabla_{\nu'} \sigma$, $\sigma_{\mu\nu} = \nabla_\mu \nabla_\nu \sigma$, etc. Let $\gamma^{\mu\nu'}$ be the matrix inverse to the matrix $\sigma_{\nu'\mu}$. Recall the identities satisfied by these matrices [2]

$$\sigma = \frac{1}{2} \sigma^\mu \sigma_\mu = \frac{1}{2} \sigma^{\mu'} \sigma_{\mu'} \quad (5.1)$$

$$\sigma_{\nu'\mu} \sigma^\mu = \sigma_{\nu'}, \quad \sigma_{\nu'\mu} \sigma^{\mu'} = \sigma_\mu, \quad (5.2)$$

$$\gamma^{\mu\nu'} \sigma_\mu = \sigma^{\nu'}, \quad \gamma^{\mu\nu'} \sigma_{\nu'} = \sigma^\mu. \quad (5.3)$$

The determinant of the matrix $\sigma_{\mu\nu'}$ defines the Van Fleck-Morette determinant [11, 2]

$$\Delta(x, x') = g^{-1/2}(x) g^{-1/2}(x') \det(-\sigma_{\mu\nu'}(x, x')). \quad (5.4)$$

It is convenient to work with the function

$$\zeta = \frac{1}{2} \log \Delta. \quad (5.5)$$

It satisfies the equation

$$\sigma^\mu \nabla_\mu \zeta = \frac{1}{2} (n - \sigma^\mu{}_\mu). \quad (5.6)$$

We also introduce the operator of parallel transport $\mathcal{P}(x, x')$ of sections along the geodesic from the point x' to the point x . It satisfies the equation

$$\sigma^\mu \nabla_\mu \mathcal{P} = 0. \quad (5.7)$$

It has the obvious properties

$$\mathcal{P}^*(x, x') = \mathcal{P}^{-1}(x, x') = \mathcal{P}(x', x). \quad (5.8)$$

We need to compute the mixed derivative of the heat kernel. Since we want to study the asymptotics as $t \rightarrow 0$ we present it in the following form [11, 2, 1]. We fix a point x' and consider a geodesic ball $B_r(x')$ centered at x' of radius r less than the injectivity radius $r_{\text{inj}}(M)$ of the manifold, $r < r_{\text{inj}}(M)$. Then in this ball the heat kernel can be presented in the form

$$U(t; x, x') = (4\pi t)^{-n/2} \exp \left\{ -\frac{\sigma(x, x')}{2t} \right\} \Psi(t; x, x'), \quad (5.9)$$

where

$$\Psi(t; x, x') = \exp \{ \zeta(x, x') \} \mathcal{P}(x, x') \Omega(t; x, x'). \quad (5.10)$$

Here Ω is the so-called transfer function that has the following asymptotic expansion as $t \rightarrow 0$

$$\Omega(t; x, x') \sim \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} a_k(x, x'), \quad (5.11)$$

where a_k are the so-called off-diagonal heat kernel coefficients. In other words, the function Ψ has the asymptotic expansion

$$\Psi(t; x, x') \sim \sum_{k=0}^{\infty} t^k \psi_k(x, x'), \quad (5.12)$$

where

$$\psi_k = \frac{(-1)^k}{k!} e^\zeta \mathcal{P} a_k. \quad (5.13)$$

By using the asymptotics of the heat kernel it is easy to obtain the asymptotics of the heat trace invariant as $t \rightarrow 0$

$$\Theta(t) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} A_k, \quad (5.14)$$

where

$$A_k = \int_M dv \operatorname{tr} [a_k], \quad (5.15)$$

where $[a_k]$ are the diagonal values of the heat kernel coefficients. The first three coefficients are (see [14, 1, 2])

$$A_0 = N \operatorname{vol}(M), \quad (5.16)$$

$$A_1 = \int_M dv \left(\operatorname{tr} Q - \frac{N}{6} R \right), \quad (5.17)$$

$$\begin{aligned} A_2 = \int_M dv \left\{ \operatorname{tr} \left(Q^2 - \frac{1}{3} Q R + \frac{1}{6} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} \right) \right. \\ \left. + N \left(\frac{1}{36} R^2 - \frac{1}{90} R_{\mu\nu} R^{\mu\nu} + \frac{1}{90} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \right) \right\}. \end{aligned} \quad (5.18)$$

It is pretty easy to compute the asymptotic expansion of the heat content $\Pi(t)$ for scalar operators on manifolds without boundary. First, note that

$$\int_M dv' U(t; x, x') = (\exp(-tL) \cdot 1)(x). \quad (5.19)$$

Therefore, the expansion of the heat content as $t \rightarrow 0$ has the form

$$\Pi(t) = \sum_{k=1}^{\infty} \frac{(-t)^k}{k!} \Pi_k, \quad (5.20)$$

where

$$\Pi_0 = \operatorname{vol} M, \quad (5.21)$$

$$\Pi_1 = \int_M dv Q, \quad (5.22)$$

$$\Pi_k = \int_M dv L^k \cdot 1 = \int_M dv Q (-\Delta + Q)^{k-2} Q, \quad k \geq 2. \quad (5.23)$$

Notice that for pure Laplacian, when $Q = 0$, the heat content is constant and is equal to its value at $t = 0$, $\Pi(t) = \text{vol}(M)$.

6 Mixed Derivative of the Heat Kernel

Now, we compute

$$\begin{aligned} \nabla_\mu \nabla_{\nu'} U &= (4\pi t)^{-n/2} \exp\left(-\frac{\sigma}{2t}\right) \left\{ \frac{1}{4t^2} \sigma_\mu \sigma_{\nu'} \Psi \right. \\ &\quad \left. - \frac{1}{2t} \left(\sigma_{\mu\nu'} \Psi + \sigma_\mu \Psi_{;\nu'} + \sigma_{\nu'} \Psi_{;\mu} \right) + \Psi_{;\mu\nu'} \right\}. \end{aligned} \quad (6.1)$$

The derivatives of the function Ψ read

$$\Psi_{;\mu} = e^\zeta \left\{ \left(\zeta_{;\mu} \mathcal{P} + \mathcal{P}_{;\mu} \right) \Omega + \mathcal{P} \Omega_{;\mu} \right\}, \quad (6.2)$$

$$\Psi_{;\mu'} = e^\zeta \left\{ \left(\zeta_{;\mu'} \mathcal{P} + \mathcal{P}_{;\mu'} \right) \Omega + \mathcal{P} \Omega_{;\mu'} \right\}, \quad (6.3)$$

$$\begin{aligned} \Psi_{;\mu\nu'} &= e^\zeta \left\{ \left(\zeta_{;\mu\nu'} \mathcal{P} + \zeta_{;\mu} \zeta_{;\nu'} \mathcal{P} + \zeta_{;\mu} \mathcal{P}_{;\nu'} + \zeta_{;\nu'} \mathcal{P}_{;\mu} + \mathcal{P}_{;\mu\nu'} \right) \Omega \right. \\ &\quad \left. + \zeta_{;\mu} \mathcal{P} \Omega_{;\nu'} + \zeta_{;\nu'} \mathcal{P} \Omega_{;\mu} + \mathcal{P}_{;\mu} \Omega_{;\nu'} + \mathcal{P}_{;\nu'} \Omega_{;\mu} + \mathcal{P} \Omega_{;\mu\nu'} \right\}. \end{aligned} \quad (6.4)$$

Therefore, we can write it in the form

$$P_{\mu\nu'} = \text{tr } U^* \nabla_\mu \nabla_{\nu'} U = \frac{1}{2t} (4\pi t)^{-n} \exp\left(-\frac{\sigma}{t}\right) e^{2\zeta} \Lambda(-\sigma_{\mu\alpha'}) Y^{\alpha'}_{\nu'}, \quad (6.5)$$

where

$$\Lambda = \text{tr } \Omega^* \Omega, \quad (6.6)$$

$$Y^{\alpha'}_{\nu'} = Z^{\alpha'}_{\nu'} + W^{\alpha'}_{\nu'} + V^{\alpha'}_{\nu'} + \frac{1}{2t} S^{\alpha'}_{\nu'}, \quad (6.7)$$

with

$$Z^{\alpha'}_{\nu'} = \delta^{\alpha'}_{\nu'} - 2t F^{\alpha'}_{\nu'}, \quad (6.8)$$

$$V^{\alpha'}_{\nu'} = \sigma^{\alpha'} E_{\nu'}, \quad (6.9)$$

$$W^{\alpha'}_{\nu'} = \tilde{E}^{\alpha'} \sigma_{\nu'}, \quad (6.10)$$

$$S^{\alpha'}_{\nu'} = -\sigma^{\alpha'} \sigma_{\nu'}. \quad (6.11)$$

and

$$E_{\nu'} = \Lambda^{-1} \mathcal{E}_{\nu'}, \quad (6.12)$$

$$\tilde{E}^{\alpha'} = \Lambda^{-1} \gamma^{\alpha'\mu} \tilde{\mathcal{E}}_\mu, \quad (6.13)$$

$$F^{\alpha'}_{\nu'} = \Lambda^{-1} \gamma^{\alpha'\mu} \mathcal{F}_{\mu\nu'}. \quad (6.14)$$

The needed tensors have the form

$$\mathcal{E}_\mu = \text{tr} \Omega^* \left\{ \left(\zeta_{;\mu} + \mathcal{P}^{-1} \mathcal{P}_{;\mu} \right) \Omega + \Omega_{;\mu} \right\}, \quad (6.15)$$

$$\tilde{\mathcal{E}}_{\mu'} = \text{tr} \Omega^* \left\{ \left(\zeta_{;\mu'} + \mathcal{P}^{-1} \mathcal{P}_{;\mu'} \right) \Omega + \Omega_{;\mu'} \right\}, \quad (6.16)$$

$$\begin{aligned} \mathcal{F}_{\mu\nu'} = \text{tr} \Omega^* \left\{ \left(\zeta_{;\mu\nu'} + \zeta_{;\mu} \zeta_{;\nu'} + \zeta_{;\mu} \mathcal{P}^{-1} \mathcal{P}_{;\nu'} + \zeta_{;\nu'} \mathcal{P}^{-1} \mathcal{P}_{;\mu} + \mathcal{P}^{-1} \mathcal{P}_{;\mu\nu'} \right) \Omega \right. \\ \left. + \zeta_{;\mu} \Omega_{;\nu'} + \zeta_{;\nu'} \Omega_{;\mu} + \mathcal{P}^{-1} \mathcal{P}_{;\mu} \Omega_{;\nu'} + \mathcal{P}^{-1} \mathcal{P}_{;\nu'} \Omega_{;\mu} + \Omega_{;\mu\nu'} \right\}. \end{aligned} \quad (6.17)$$

Now, we need to compute the determinant of the matrix $P = P_{\mu\nu'}$. We obtain

$$\det P = g^{1/2}(x) g^{1/2}(x') (2t)^{-n} (4\pi t)^{-n^2} \exp\left(-n \frac{\sigma}{t}\right) \Delta H, \quad (6.18)$$

where

$$H = e^{2n\zeta} \Lambda^n \det Y^{\alpha'}_{\nu'}. \quad (6.19)$$

We will use the following formula for the determinant of the sum of two matrices. By using the following formula for the determinant of the matrix $I + C$,

$$\begin{aligned} \det(I + C) &= \sum_{k=0}^n C^{\alpha_1}_{[\alpha_1} \cdots C^{\alpha_k}_{\alpha_k]} \\ &= 1 + \text{tr} C + \frac{1}{2} (\text{tr} C)^2 - \frac{1}{2} \text{tr} C^2 + \cdots + C^{\alpha_1}_{[\alpha_1} \cdots C^{\alpha_n}_{\alpha_n]} \end{aligned} \quad (6.20)$$

we obtain the determinant of the sum of two matrices $A + B$ (with B being an invertible matrix)

$$\det(A + B) = \det B \sum_{k=0}^n A^{\beta_1}_{[\alpha_1} \cdots A^{\beta_k}_{\alpha_k]} B^{-1\alpha_1}_{\beta_1} \cdots B^{-1\alpha_k}_{\beta_k}. \quad (6.21)$$

Let $X = (X^{\nu'}_{\mu'})$ be the inverse of the matrix $Z = (Z^{\mu'}_{\nu'})$ and

$$J = \det Z. \quad (6.22)$$

Next, we note that

$$S^{[\mu_1]_{[\nu'_1]} S^{\mu_2]_{\nu'_2]}} = V^{[\mu_1]_{[\nu'_1]} V^{\mu_2]_{\nu'_2]}} = W^{[\mu_1]_{[\nu'_1]} W^{\mu_2]_{\nu'_2]}} = S^{[\mu_1]_{[\nu'_1]} V^{\mu_2]_{\nu'_2]}} = S^{[\mu_1]_{[\nu'_1]} W^{\mu_2]_{\nu'_2]}} = 0. \quad (6.23)$$

Then by using the above formula (6.21) for the determinant we obtain

$$H = J e^{2n\zeta} \Lambda^n \left(1 + V^{\mu'}_{\nu'} X^{\nu'}_{\mu'} + W^{\mu'}_{\nu'} X^{\nu'}_{\mu'} + \frac{1}{2t} S^{\mu'}_{\nu'} X^{\nu'}_{\mu'} + 2V^{[\mu'}_{\alpha'} W^{\nu']_{\beta'}} X^{\alpha'}_{\mu'} X^{\beta'}_{\nu'} \right) \quad (6.24)$$

This can be expressed in terms of a few invariants; let

$$\chi_1 = \sigma_{\beta'} X^{\beta'}_{\nu'} \sigma^{\nu'}, \quad (6.25)$$

$$\chi_2 = E_{\nu'} X^{\nu'}_{\mu'} \sigma^{\mu'}, \quad (6.26)$$

$$\chi_3 = \sigma_{\nu'} X^{\nu'}_{\mu'} \tilde{E}^{\mu'}, \quad (6.27)$$

$$\chi_4 = E_{\alpha'} X^{\alpha'}_{\mu'} \tilde{E}^{\mu'}. \quad (6.28)$$

Then

$$H = J e^{2n\zeta} \Lambda^n \left(-\frac{1}{2t} \chi_1 + 1 + \chi_2 + \chi_3 + \chi_2 \chi_3 - \chi_1 \chi_4 \right). \quad (6.29)$$

7 Heat Determinant Asymptotics

We are studying the asymptotics as $t \rightarrow 0$ of the functional $K(t)$. As we mentioned above we fix a point x' in the manifold M and consider a geodesic ball $B_r(x')$ centered at x' of radius $r < r_{\text{inj}}(M)$ smaller than the injectivity radius $r_{\text{inj}}(M)$ of the manifold. We decompose the integral defining the invariant $K(t)$ in two parts

$$K(t) = K_{\text{diag}}(t) + K_{\text{off-diag}}(t), \quad (7.1)$$

where

$$K_{\text{diag}}(t) = \int_M dx' \int_{B_r(x')} dx \det P_{\mu\nu'}(t; x, x'), \quad (7.2)$$

$$K_{\text{off-diag}}(t) = \int_M dx' \int_{M-B_r(x')} dx \det P_{\mu\nu'}(t; x, x'). \quad (7.3)$$

By using the standard elliptic estimates of the heat kernel [15] one can show that for any $x \in M - B_r(x')$ and $0 < t < 1$ there is an estimate

$$|U(t; x, x')| \leq C_1 t^{-n/2} \exp\left(-\frac{r^2}{4t}\right), \quad (7.4)$$

and similarly

$$|P_{\mu\nu'}(t; x, x')| \leq C_2 t^{-n-2} \exp\left(-\frac{r^2}{2t}\right), \quad (7.5)$$

where C_1 and C_2 are constants; therefore,

$$|\det P_{\mu\nu'}(t; x, x')| \leq C_2 t^{-n(n+2)} \exp\left(-\frac{nr^2}{2t}\right), \quad (7.6)$$

and

$$|K_{\text{off-diag}}(t)| \leq C_3 t^{-n(n+2)} \exp\left(-\frac{nr^2}{2t}\right), \quad (7.7)$$

where C_3 is another constant.

Thus, we see that the off-diagonal part $K_{\text{off-diag}}(t)$ is exponentially small as $t \rightarrow 0$ and does not contribute to the asymptotic expansion of the invariant $K(t)$ as $t \rightarrow 0$, that is, as $t \rightarrow 0$

$$K(t) \sim K_{\text{diag}}(t), \quad (7.8)$$

and, hence,

$$K(t) \sim 2^{-n} (4\pi)^{-n^2} t^{-n(n+1)} \int_M dv' \int_{B_r(x')} dv \exp\left(-n \frac{\sigma(x, x')}{t}\right) \Delta(x, x') H(t; x, x'). \quad (7.9)$$

To compute this integral as $t \rightarrow 0$ we make the change of variables $x^\mu \mapsto \xi^{\nu'}$ described below. It is convenient to introduce new coordinates

$$\xi^{\nu'} = \frac{\sigma^{\nu'}}{\sqrt{t}}, \quad (7.10)$$

Note that

$$|\xi|^2 = \xi^{\nu'} \xi_{\nu'} = \frac{2\sigma}{t}. \quad (7.11)$$

The volume element changes as follows

$$dv(x) = g^{1/2}(x) dx = t^{n/2} \Delta^{-1}(x, x') g^{1/2}(x') d\xi. \quad (7.12)$$

The integration over ξ goes over the Euclidean ball $B_{r/\sqrt{t}}(0)$ of radius r/\sqrt{t} , that is, $|\xi| \leq r/\sqrt{t}$. As $t \rightarrow 0$ the radius of the ball goes to infinity, therefore, the asymptotics $t \rightarrow 0$ are determined by the integral over the whole Euclidean space \mathbb{R}^n . Alternatively, we could split the integral over the ball $B_{r/\sqrt{t}}(0)$ as the integral

over the whole \mathbb{R}^n minus the integral over $\mathbb{R}^n - B_{r/\sqrt{t}}(0)$ and then show that the second integral is exponentially small as $t \rightarrow 0$.

Thus, we get

$$K(t) \sim 2^{-n} (4\pi)^{-n^2} t^{-n(n+\frac{1}{2})} \times \int_M dv' \int_{\mathbb{R}^n} d\xi g^{1/2}(x') \exp\left(-\frac{n}{2}|\xi|^2\right) H(t; x, x'). \quad (7.13)$$

This can be rewritten in the form

$$K(t) \sim (4\pi)^{-n^2} \left(\frac{\pi}{2n}\right)^{n/2} t^{-n(n+\frac{1}{2})} \int_M dv' \langle H \rangle, \quad (7.14)$$

where the brackets $\langle \dots \rangle$ denote the Gaussian average over the variables ξ defined by

$$\langle f(\xi) \rangle = \left(\frac{n}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} d\xi g^{1/2}(x') \exp\left(-\frac{n}{2}|\xi|^2\right) f(\xi). \quad (7.15)$$

By using the asymptotic expansion (5.12) one can show that all matrices Z, V, W, S introduced above have asymptotic expansions in non-negative integer powers of t ; therefore, there exists an expansion

$$H(t; x, x') \sim \sum_{k=-1}^{\infty} t^k h_k(x, x'). \quad (7.16)$$

Therefore,

$$K(t) \sim (4\pi)^{-n^2} \left(\frac{\pi}{2n}\right)^{n/2} t^{-n(n+\frac{1}{2})} \sum_{k=-1}^{\infty} H_k(t), \quad (7.17)$$

where

$$H_k(t) = \int_M dv' \langle h_k \rangle. \quad (7.18)$$

To compute the Gaussian average we expand the coefficients h_k in covariant Taylor series at the point x'

$$\begin{aligned} h_k(x, x') &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sigma^{\mu'_1} \dots \sigma^{\mu'_m} h_{k, \mu'_1 \dots \mu'_m}(x') \\ &= \sum_{m=0}^{\infty} t^{m/2} \frac{(-1)^m}{m!} \xi^{\mu'_1} \dots \xi^{\mu'_m} h_{k, \mu'_1 \dots \mu'_m}(x'), \end{aligned} \quad (7.19)$$

where

$$h_{k,\mu'_1\ldots\mu'_m}(x') = \left[\nabla_{(\mu_1} \cdots \nabla_{\mu_m)} h_k(x, x') \right]_{x=x'} . \quad (7.20)$$

The Gaussian averages of the monomials are well known

$$\langle \xi^{\mu_1} \cdots \xi^{\mu_{2k+1}} \rangle = 0 , \quad (7.21)$$

$$\langle \xi^{\mu_1} \cdots \xi^{\mu_{2k}} \rangle = \frac{(2k)!}{(2n)^k k!} g^{(\mu_1 \mu_2} \cdots g^{\mu_{2k-1} \mu_{2k})} . \quad (7.22)$$

In particular,

$$\langle \xi^\mu \xi^\nu \rangle = \frac{1}{n} g^{\mu\nu} , \quad (7.23)$$

$$\langle \xi^\mu \xi^\nu \xi^\alpha \xi^\beta \rangle = \frac{1}{n^2} \left(g^{\mu\nu} g^{\alpha\beta} + g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} \right) . \quad (7.24)$$

This enables us to immediately compute the Gaussian average

$$\langle h_k \rangle = \sum_{j=0}^{\infty} t^j h_{k,j} , \quad (7.25)$$

where

$$h_{k,j} = \frac{1}{(2n)^j j!} g^{\mu_1 \mu_2} \cdots g^{\mu_{2j-1} \mu_{2j}} h_{k,\mu'_1 \cdots \mu'_{2j}} . \quad (7.26)$$

So,

$$H_k(t) = \sum_{j=0}^{\infty} t^j H_{k,j} , \quad (7.27)$$

where

$$H_{k,j} = \int_M dv h_{k,j} . \quad (7.28)$$

Therefore, the heat determinant has the asymptotics as $t \rightarrow 0$

$$K(t) \sim (4\pi)^{-n^2} \left(\frac{\pi}{2n} \right)^{n/2} t^{-n(n+\frac{1}{2})} \sum_{k=-1}^{\infty} t^k B_k , \quad (7.29)$$

where

$$B_k = \int_M dv b_k = \sum_{j=-1}^k H_{j,k-j} , \quad (7.30)$$

with

$$b_k = \sum_{j=-1}^k h_{j,k-j}. \quad (7.31)$$

The coefficients B_k are the new heat invariants that are of central interest of this paper. They are not spectral invariants since they depend on the eigenfunctions as well. They are invariants built from the curvatures and their derivatives (as well as the potential term Q) with numerical coefficients that are universal since they depend only on the dimension of the manifold and the dimension of the vector bundle. Their calculation is reduced to the calculation of the coefficients $H_{k,m}$, which, in turn, are determined by the Taylor coefficients $h_{k,\mu'_1 \dots \mu'_{2m}}$ of the coefficients h_k .

8 Calculation of Low Order Coefficients

8.1 Calculation of the Coefficients h_{-1}, h_0, h_1, h_2

We will compute the first four coefficients

$$b_{-1} = h_{-1,0}, \quad (8.1)$$

$$b_0 = h_{0,0} + h_{-1,1}, \quad (8.2)$$

$$b_1 = h_{1,0} + h_{0,1} + h_{-1,2}, \quad (8.3)$$

$$b_2 = h_{2,0} + h_{1,1} + h_{0,2} + h_{-1,3}. \quad (8.4)$$

We have

$$h_{-1,0} = [h_{-1}], \quad (8.5)$$

$$h_{-1,1} = \frac{1}{2n} g^{\mu\nu} h_{-1,\mu\nu}, \quad (8.6)$$

$$h_{-1,2} = \frac{1}{8n^2} g^{\mu\nu} g^{\alpha\beta} h_{-1,\mu\nu\alpha\beta}, \quad (8.7)$$

$$h_{-1,3} = \frac{1}{48n^3} g^{\mu\nu} g^{\alpha\beta} g^{\rho\sigma} h_{-1,\mu\nu\alpha\beta\rho\sigma}, \quad (8.8)$$

$$h_{0,0} = [h_0], \quad (8.9)$$

$$h_{0,1} = \frac{1}{2n} g^{\mu\nu} h_{0,\mu\nu}, \quad (8.10)$$

$$h_{0,2} = \frac{1}{8n^2} g^{\mu\nu} g^{\alpha\beta} h_{0,\mu\nu\alpha\beta}, \quad (8.11)$$

$$h_{1,0} = [h_1], \quad (8.12)$$

$$h_{1,1} = \frac{1}{2n} g^{\mu\nu} h_{1,\mu\nu}, \quad (8.13)$$

$$h_{2,0} = [h_2]. \quad (8.14)$$

Here the square brackets denote the coincidence limits, as usual.

Thus, we need to compute first the coefficients h_{-1}, h_0, h_1, h_2

$$H \sim t^{-1}h_{-1} + h_0 + th_1 + t^2h_2 + \dots \quad (8.15)$$

in terms of ψ_k and then compute the coincidence limits of their derivatives (or expand them in Taylor series). Note that we need h_{-1} up to sixth order in $\sigma^{\mu'}$, the coefficient h_0 up to the forth order in $\sigma^{\mu'}$ the coefficient h_1 up to the second order and the coefficient h_2 up to the zero order. What we actually do is introduce a small parameter ε so that

$$t \sim \varepsilon^2, \quad \sigma^{\mu'} \sim \varepsilon. \quad (8.16)$$

Since we want to compute the terms of the order t^2 we need to keep the terms of the order ε^4 and neglect the terms of higher order in ε . All expansions below are valid in this approximation. The dots below denote the neglected terms of order $O(\varepsilon^5)$.

We have

$$H = J e^{2n\zeta} \Lambda^n \Phi, \quad (8.17)$$

where

$$\Phi = -\frac{1}{2t}\chi_1 + 1 + \chi_2 + \chi_3 + \chi_2\chi_3 - \chi_1\chi_4. \quad (8.18)$$

The expansion of the function Λ has the form

$$\Lambda = N + t\Lambda_1 + t^2\Lambda_2 + \dots, \quad (8.19)$$

so that

$$\Lambda^{-1} = \frac{1}{N} - \frac{1}{N^2}t\Lambda_1 + t^2\left(\frac{1}{N^3}\Lambda_1^2 - \frac{1}{N^2}\Lambda_2\right) + \dots \quad (8.20)$$

and

$$\Lambda^n = N^n + tnN^{n-1}\Lambda_1 + t^2\left(nN^{n-1}\Lambda_2 + \frac{n(n-1)}{2}N^{n-2}\Lambda_1^2\right) + \dots, \quad (8.21)$$

where

$$\Lambda_1 = -\text{tr}(a_1 + a_1^*), \quad (8.22)$$

$$\Lambda_2 = \frac{1}{2}\text{tr}(a_2 + a_2^* + 2a_1a_1^*). \quad (8.23)$$

To compute the expansion of needed functions we will need to find the expansion of the tensors E , \tilde{E} and F (6.12)-(6.14). We will compute them later. For now, let us denote them by

$$E_{\nu'} = E_{0,\nu'} + tE_{1,\nu'} + t^2E_{2,\nu'} + \dots, \quad (8.24)$$

$$\tilde{E}^{\nu'} = \tilde{E}_0^{\nu'} + t\tilde{E}_1^{\nu'} + t^2\tilde{E}_2^{\nu'} + \dots, \quad (8.25)$$

$$F^{\alpha'}_{\nu'} = F_0^{\alpha'}_{\nu'} + tF_1^{\alpha'}_{\nu'} + t^2F_2^{\alpha'}_{\nu'} + \dots, \quad (8.26)$$

Then the expansion of the function J (6.22), the determinant of the matrix Z (6.8), has the form

$$J = 1 + tJ_1 + t^2J_2 + \dots, \quad (8.27)$$

where

$$J_1 = -2F_0^{\mu'}_{\mu'}, \quad (8.28)$$

$$J_2 = 2F_0^{\mu'}_{\mu'}F_0^{\nu'}_{\nu'} - 2F_0^{\mu'}_{\nu'}F_0^{\nu'}_{\mu'} - 2F_1^{\mu'}_{\mu'}. \quad (8.29)$$

The expansion of the matrix X , the inverse of the matrix Z (6.8), reads

$$X^{\alpha'}_{\nu'} = \delta^{\alpha'}_{\nu'} + tX_1^{\alpha'}_{\nu'} + t^2X_2^{\alpha'}_{\nu'} + \dots, \quad (8.30)$$

where

$$X_1^{\alpha'}_{\nu'} = 2F_0^{\alpha'}_{\nu'}, \quad (8.31)$$

$$X_2^{\alpha'}_{\nu'} = 4F_0^{\alpha'}_{\mu'}F_0^{\mu'}_{\nu'} + 2F_1^{\alpha'}_{\nu'}. \quad (8.32)$$

We need to compute the expansion of the function Φ up to terms of order $O(\varepsilon^5)$ (recall that $t \sim \varepsilon^2$ and $\sigma^{\mu'} \sim \varepsilon$). Note that

$$\chi_1 \sim O(\varepsilon^2), \quad \chi_2, \chi_3 \sim O(\varepsilon), \quad \chi_4 \sim O(1). \quad (8.33)$$

Therefore, since χ_1 is divided by t we need χ_1 up to terms of order t^2 . Also, we only need to keep terms up to order t in χ_2 and χ_3 . Further, since the function χ_4 comes only with the product with the function χ_1 , we need the function χ_4 also only up to linear terms in t , that is

$$\chi_1 = \chi_{1,0} + t\chi_{1,1} + t^2\chi_{1,2} + \dots, \quad (8.34)$$

$$\chi_2 = \chi_{2,0} + t\chi_{2,1} + \dots, \quad (8.35)$$

$$\chi_3 = \chi_{3,0} + t\chi_{3,1} + \dots, \quad (8.36)$$

$$\chi_4 = \chi_{4,0} + t\chi_{4,1} + \dots. \quad (8.37)$$

By using the expansion of the matrix X and the quantities E , \tilde{E} and F we obtain

$$\chi_{1,0} = 2\sigma, \quad (8.38)$$

$$\chi_{1,1} = 2\sigma_{\nu'} F_{0\mu'}^{\nu'} \sigma^{\mu'}, \quad (8.39)$$

$$\chi_{1,2} = 2\sigma_{\nu'} F_{1\mu'}^{\nu'} \sigma^{\mu'} + 4\sigma_{\nu'} F_{0\alpha'}^{\nu'} F_{0\mu'}^{\alpha'} \sigma^{\mu'}, \quad (8.40)$$

$$\chi_{2,0} = E_{0,\mu'} \sigma^{\mu'}, \quad (8.41)$$

$$\chi_{2,1} = 2E_{1,\mu'} \sigma^{\mu'} + 2E_{0,\nu'} F_{0\mu'}^{\nu'} \sigma^{\mu'}, \quad (8.42)$$

$$\chi_{3,0} = \sigma_{\mu'} \tilde{E}_0^{\mu'}, \quad (8.43)$$

$$\chi_{3,1} = 2\sigma_{\nu'} \tilde{E}_1^{\mu'} + 2\sigma_{\nu'} F_{0\mu'}^{\nu'} \tilde{E}_0^{\mu'}, \quad (8.44)$$

$$\chi_{4,0} = E_{0,\mu'} \tilde{E}_0^{\mu'}. \quad (8.45)$$

$$\chi_{4,1} = E_{0,\mu'} \tilde{E}_1^{\mu'} + E_{1,\mu'} \tilde{E}_0^{\mu'} + 2E_{0,\nu'} F_{0\mu'}^{\nu'} \tilde{E}_0^{\mu'}. \quad (8.46)$$

By the same reason we only need the expansion of the function Φ up to linear terms in t (we neglect the terms of order higher than $O(\varepsilon^4)$; the quadratic terms in t will be of order $O(\varepsilon^5)$). We get

$$\Phi = -\frac{\sigma}{t} + \Phi_0 + t\Phi_1 + \dots, \quad (8.47)$$

where

$$\Phi_0 = 1 - \frac{1}{2}\chi_{1,1} + \chi_{2,0} + \chi_{3,0} + \chi_{2,0}\chi_{3,0} - \chi_{1,0}\chi_{4,0}, \quad (8.48)$$

$$\Phi_1 = -\frac{1}{2}\chi_{1,2} + \chi_{2,1} + \chi_{3,1} + \chi_{2,0}\chi_{3,1} + \chi_{2,1}\chi_{3,0} - \chi_{1,1}\chi_{4,0} - \chi_{1,0}\chi_{4,1}. \quad (8.49)$$

This gives the coefficients h_k of the expansion of the function H (in the needed order in ε)

$$h_{-1} = -N^n \sigma e^{2n\zeta}, \quad (8.50)$$

$$h_0 = e^{2n\zeta} \left\{ N^n \Phi_0 - N^n \sigma J_1 - nN^{n-1} \sigma \Lambda_1 \right\}, \quad (8.51)$$

$$h_1 = e^{2n\zeta} \left\{ N^n J_1 \Phi_0 + nN^{n-1} \Lambda_1 \Phi_0 + N^n \Phi_1 - \sigma \left(N^n J_2 + nN^{n-1} \Lambda_2 + \frac{n(n-1)}{2} N^{n-2} \Lambda_1^2 + nN^{n-1} J_1 \Lambda_1 \right) \right\}, \quad (8.52)$$

$$h_2 = e^{2n\zeta} \left\{ N^n J_1 \Phi_1 + nN^{n-1} \Lambda_1 \Phi_1 + \Phi_0 \left(N^n J_2 + nN^{n-1} \Lambda_2 + \frac{n(n-1)}{2} N^{n-2} \Lambda_1^2 + nN^{n-1} J_1 \Lambda_1 \right) \right\}. \quad (8.53)$$

8.2 Calculation of b_{-1}

By using the explicit form of the coefficient h_{-1} we see that its coincidence limit vanishes,

$$h_{-1,0} = [h_{-1}] = 0. \quad (8.54)$$

Thus, the coefficient b_{-1} vanishes as well

$$b_{-1} = 0. \quad (8.55)$$

8.3 Calculation of b_0

Next, by using the coefficient h_{-1} and the coincidence limits of the functions σ and ζ (see Appendix) we compute

$$h_{-1,\mu\nu} = -N^n g_{\mu\nu}. \quad (8.56)$$

Therefore,

$$h_{-1,1} = -\frac{1}{2}N^n. \quad (8.57)$$

Next, we compute

$$h_{0,0} = N^n [\Phi_0], \quad (8.58)$$

It is easy to see that

$$[\Phi_0] = 1, \quad (8.59)$$

and, therefore,

$$h_{0,0} = N^n. \quad (8.60)$$

This gives the coefficient b_0

$$b_0 = \frac{1}{2}N^n. \quad (8.61)$$

8.4 Calculation of b_1

First, by using the derivatives of the function ζ we obtain

$$h_{-1,\mu\nu\alpha\beta} = -\frac{n}{3}N^n g_{(\mu\nu} R_{\alpha\beta)}. \quad (8.62)$$

By contracting all indices we get

$$h_{-1,2} = -N^n \frac{n+2}{72n} R. \quad (8.63)$$

Then we compute

$$h_{1,0} = N^n[J_1] + nN^{n-1}[\Lambda_1] + N^n[\Phi_1]. \quad (8.64)$$

We have

$$[\Lambda_1] = -\text{tr}[a_1 + a_1^*] = -2\text{tr } Q + \frac{1}{3}NR \quad (8.65)$$

and

$$[\Phi_1] = 0. \quad (8.66)$$

To compute

$$[J_1] = -2[F_0^{\mu'}{}_{\mu'}], \quad (8.67)$$

we need to compute the coincidence limit $[F_0^{\mu'}{}_{\mu'}]$. By using the coincidence limits of the derivatives of the functions ζ and \mathcal{P} we obtain

$$[F_0^{\alpha'}{}_{\nu'}] = \frac{1}{6}R^\alpha{}_\nu. \quad (8.68)$$

Therefore,

$$[J_1] = -\frac{1}{3}R \quad (8.69)$$

and

$$h_{1,0} = N^n \frac{(n-1)}{3}R - 2nN^{n-1}\text{tr } Q. \quad (8.70)$$

Now we need to compute $h_{0,1}$. First, we have

$$h_{0,\mu\nu} = 2nN^n[\zeta_{;\mu\nu}] + N^n[\Phi_{0;(\mu\nu)}] - N^n[J_1]g_{\mu\nu} - nN^{n-1}[\Lambda_1]g_{\mu\nu}. \quad (8.71)$$

Thus we need to compute $[\Phi_{0;(\mu\nu)}]$. By using the definition of Φ_0 and the functions χ_k we get

$$[\Phi_{0;(\mu\nu)}] = \left[-\frac{1}{2}\chi_{1,1;(\mu\nu)} + \chi_{2,0;(\mu\nu)} + \chi_{3,0;(\mu\nu)} + 2\chi_{2,0;(\mu}\chi_{3,0;\nu)} - \chi_{1,0;(\mu\nu)}\chi_{4,0} \right]. \quad (8.72)$$

This gives

$$[\chi_{1,0;(\mu\nu)}] = 2g_{\mu\nu}, \quad (8.73)$$

$$[\chi_{1,1;(\mu\nu)}] = [4F_{0,(\mu'}{}_{\nu')}] = \frac{2}{3}R_{\mu\nu}, \quad (8.74)$$

$$[\chi_{2,0;\mu}] = -[E_{0,\mu'}], \quad (8.75)$$

$$[\chi_{3,0;\mu}] = -[\tilde{E}_{0,\mu'}], \quad (8.76)$$

$$[\chi_{2,0;(\mu\nu)}] = -2[E_{0,(\mu'}{}_{\nu)}], \quad (8.77)$$

$$[\chi_{3,0;(\mu\nu)}] = -2[\tilde{E}_{0,(\mu'}{}_{\nu)}], \quad (8.78)$$

$$[\chi_{4,0}] = [E_{0,\mu'}\tilde{E}_0^{\mu'}]. \quad (8.79)$$

Now, we need to compute the coincidence limits of the functions E_0 and \tilde{E}_0 and their derivatives. It is easy to see that

$$[E_{0,\mu'}] = [\tilde{E}_{0,\mu'}] = 0. \quad (8.80)$$

So,

$$[\chi_{2,0,\mu}] = [\chi_{3,0,\mu}] = [\chi_{4,0}] = 0. \quad (8.81)$$

Therefore,

$$[\Phi_{0;(\mu\nu)}] = \left[-\frac{1}{2}\chi_{1,1;(\mu\nu)} + \chi_{2,0;(\mu\nu)} + \chi_{3,0;(\mu\nu)} \right]. \quad (8.82)$$

Next, by using the coincidence limits of the functions ζ and \mathcal{P} we get

$$[\tilde{E}_{0,(\mu';\nu)}] = -[\zeta_{\mu\nu}] = -\frac{1}{6}R_{\mu\nu}, \quad (8.83)$$

$$[E_{0,(\mu';\nu)}] = [\zeta_{(\mu'\nu)}] = -\frac{1}{6}R_{\mu\nu}. \quad (8.84)$$

This gives

$$[\Phi_{0;(\mu\nu)}] = \frac{1}{3}R_{\mu\nu}. \quad (8.85)$$

Therefore,

$$h_{0,\mu\nu} = N^n \frac{n+1}{3} R_{\mu\nu} + N^n \frac{1-n}{3} R g_{\mu\nu} + 2nN^{n-1} g_{\mu\nu} \text{tr } Q. \quad (8.86)$$

Thus, by contracting the indices we get

$$h_{0,1} = \frac{1}{2n} g^{\mu\nu} h_{0,\mu\nu} = N^n \frac{1+2n-n^2}{6n} R + nN^{n-1} \text{tr } Q. \quad (8.87)$$

Finally, by collecting all terms we get

$$b_1 = N^n \frac{12n^2 - n + 10}{72n} R - nN^{n-1} \text{tr } Q. \quad (8.88)$$

8.5 Calculation of b_2

To compute the coefficient b_2 we need to compute $h_{-1,3}$, $h_{0,2}$, $h_{1,1}$ and $h_{2,0}$.

First, we compute

$$h_{-1,\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} = -30nN^n \left(\zeta_{(\mu_1\mu_2\mu_3\mu_4} g_{\mu_5\mu_6)} + 6n\zeta_{(\mu_1\mu_2} \zeta_{\mu_3\mu_4} g_{\mu_5\mu_6)} \right). \quad (8.89)$$

Next, we need to contract all indices. One can show that for any two symmetric tensors A and B

$$g^{\mu_1\mu_2} g^{\mu_3\mu_4} g^{\mu_5\mu_6} A_{(\mu_1\mu_2\mu_3\mu_4} B_{\mu_5\mu_6)} = \frac{1}{15} (3A^{\mu\nu}{}_{\mu\nu} B^\alpha{}_\alpha + 12A^\alpha{}_{\alpha\mu\nu} B^{\mu\nu}). \quad (8.90)$$

Similarly, we can show that for any symmetric tensor C we have

$$g^{\mu_1\mu_2} C_{(\mu_1\mu_2} C_{\mu_3\mu_4)} = \frac{1}{3} (C^\mu{}_\mu C_{\mu_3\mu_4} + 2C^\alpha{}_{(\mu_3} C_{\mu_4)\alpha}). \quad (8.91)$$

By using these two equations we show that for any two symmetric tensors C and B

$$\begin{aligned} & g^{\mu_1\mu_2} g^{\mu_3\mu_4} g^{\mu_5\mu_6} C_{(\mu_1\mu_2} C_{\mu_3\mu_4} B_{\mu_5\mu_6)} \\ &= \frac{1}{15} \left\{ C^\mu{}_\mu C^\nu{}_\nu B^\alpha{}_\alpha + 2C^{\mu\nu} C_{\mu\nu} B^\alpha{}_\alpha + 4C^\alpha{}_\alpha C_{\mu\nu} B^{\mu\nu} + 8C^\alpha{}_\mu C_{\alpha\nu} B^{\mu\nu} \right\}. \end{aligned} \quad (8.92)$$

This enables us to compute

$$g^{\mu_1\mu_2} g^{\mu_3\mu_4} g^{\mu_5\mu_6} \zeta_{(\mu_1\mu_2\mu_3\mu_4} g_{\mu_5\mu_6)} = \frac{n+4}{5} \zeta_{(4)}, \quad (8.93)$$

where

$$\zeta_{(4)} = g^{\mu_1\mu_2} g^{\mu_3\mu_4} \zeta_{(\mu_1\mu_2\mu_3\mu_4)}, \quad (8.94)$$

and

$$g^{\mu_1\mu_2} g^{\mu_3\mu_4} g^{\mu_5\mu_6} \zeta_{(\mu_1\mu_2} \zeta_{\mu_3\mu_4} g_{\mu_5\mu_6)} = \frac{n+4}{15} (\zeta^\mu{}_\mu \zeta^\nu{}_\nu + 2\zeta^{\mu\nu} \zeta_{\mu\nu}). \quad (8.95)$$

Thus, we get

$$h_{-1,3} = -N^n \frac{(n+4)}{8n^2} \left[\zeta_{(4)} + 2n \zeta^\mu{}_\mu \zeta^\nu{}_\nu + 4n \zeta^{\mu\nu} \zeta_{\mu\nu} \right]. \quad (8.96)$$

The tensors $[\zeta_{\mu\nu}]$ and $[\zeta_{(\mu\nu\alpha\beta)}]$ are listed in the Appendix. By using these tensors we obtain

$$h_{-1,3} = -N^n \frac{(n+4)}{8n^2} \left(\frac{1}{5} \nabla^\mu \nabla^\mu R + \frac{n}{18} R^2 + \frac{5n+1}{45} R_{\mu\nu} R^{\mu\nu} + \frac{1}{30} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \right). \quad (8.97)$$

Next, we compute $h_{2,0}$. We already have

$$[\Phi_1] = 0, \quad (8.98)$$

$$[J_1] = -\frac{1}{3}R, \quad (8.99)$$

$$[\Lambda_1] = \frac{1}{3}NR - 2\text{tr } Q, \quad (8.100)$$

$$[\Lambda_2] = \text{tr } [a_2] + \text{tr } [a_1]^2. \quad (8.101)$$

We compute $[J_2]$. By using F_0 we get

$$[J_2] = \frac{1}{18}R^2 - \frac{1}{18}R_{\mu\nu}R^{\mu\nu} - 2[F_{1,\mu'}^{\mu'}]. \quad (8.102)$$

Next, we compute the matrix F_1 ,

$$[F_{1,\nu'}^{\mu'}] = \frac{1}{N}\text{tr } [\nabla_{\nu'}\nabla^{\mu'}a_1] + \frac{2}{N}\text{tr } [\mathcal{P}^{\mu'}_{\nu'}]Q. \quad (8.103)$$

The contraction of this matrix reads

$$[F_{1,\mu'}^{\mu'}] = \frac{1}{N}\text{tr } [\nabla_{\mu'}\nabla^{\mu'}a_1]. \quad (8.104)$$

We use the equation

$$[\nabla^{\mu'}\nabla_{\mu}a_1] = \nabla^{\mu}[\nabla_{\mu}a_1] - [\nabla^{\mu}\nabla_{\mu}a_1]. \quad (8.105)$$

By using the equations in the appendix we obtain

$$[\nabla^{\mu'}\nabla_{\mu}a_1] = \frac{1}{6}\nabla^{\mu}\nabla_{\mu}Q + \frac{1}{6}\mathcal{R}_{\alpha\mu}\mathcal{R}^{\alpha\mu} - \frac{1}{60}\nabla^{\mu}\nabla_{\mu}R \quad (8.106)$$

$$- \frac{1}{90}R_{\mu\nu}R^{\mu\nu} + \frac{1}{90}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}. \quad (8.107)$$

This gives

$$\begin{aligned} [J_2] &= \frac{1}{30}\nabla^{\mu}\nabla_{\mu}R + \frac{1}{18}R^2 - \frac{1}{30}R_{\mu\nu}R^{\mu\nu} - \frac{1}{45}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \\ &\quad + \frac{1}{N}\text{tr} \left(-\frac{1}{3}\nabla^{\mu}\nabla_{\mu}Q - \frac{1}{3}\mathcal{R}_{\alpha\mu}\mathcal{R}^{\alpha\mu} \right). \end{aligned} \quad (8.108)$$

By collecting all terms we obtain

$$\begin{aligned}
h_{2,0} = & -\frac{n+1}{3}N^{n-1}\text{tr}\nabla^\mu\nabla_\mu Q + \frac{2n+1}{30}N^n\nabla^\mu\nabla_\mu R + \frac{n-2}{6}N^{n-1}\text{tr}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} \\
& + 2nN^{n-1}\text{tr}Q^2 + 2n(n-1)N^{n-2}(\text{tr}Q)^2 - \frac{2n(n-1)}{3}N^{n-1}R\text{tr}Q \\
& + \frac{(n-1)^2}{18}N^nR^2 - \frac{n+3}{90}N^nR_{\mu\nu}R^{\mu\nu} + \frac{n-2}{90}N^nR_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}. \quad (8.109)
\end{aligned}$$

Now we compute $h_{1,1}$. We have

$$\begin{aligned}
h_{1,\mu\nu} = & \left[N^n\Phi_{1;(\mu\nu)} + (N^nJ_1 + nN^{n-1}\Lambda_1)(\Phi_{0;(\mu\nu)} + 2n\zeta_{;\mu\nu}) \right. \\
& \left. + N^n[J_{1;\mu\nu}] + nN^{n-1}[\Lambda_{1;\mu\nu}] - g_{\mu\nu}h_{2,0} \right]. \quad (8.110)
\end{aligned}$$

Therefore,

$$\begin{aligned}
h_{1,1} = & \frac{1}{2n} \left[N^n\Phi_1{}^{;\mu}{}_\mu + [N^nJ_1 + nN^{n-1}\Lambda_1](\Phi_0{}^{;\mu}{}_\mu + 2n\zeta^{;\mu}{}_\mu) \right. \\
& \left. + N^n[J_1{}^{;\mu}{}_\mu] + nN^{n-1}[\Lambda_1{}^{;\mu}{}_\mu] - nh_{2,0} \right]. \quad (8.111)
\end{aligned}$$

We already know $[h_2]$, $[J_1]$, $[\Lambda_1]$, $[\zeta_{;\mu\nu}]$ and $[\Phi_{0;(\mu\nu)}]$. The only new objects to compute are $\Phi_{1;(\mu\nu)}$, $J_{1;(\mu\nu)}$, and $\Lambda_{1;(\mu\nu)}$. By using the definition of the functions $\chi_{i,j}$ we find that the coincidence limits of all first derivatives vanish, that is,

$$[\chi_{i,j,\mu}] = 0. \quad (8.112)$$

Therefore,

$$[\Phi_{1;(\mu\nu)}] = \left[-\frac{1}{2}\chi_{1,2;(\mu\nu)} + \chi_{2,1;(\mu\nu)} + \chi_{3,1;(\mu\nu)} \right]. \quad (8.113)$$

We compute

$$[\chi_{1,2;(\mu\nu)}] = 4[F_{1,(\mu'\nu')}] + 8[F_{0,\mu'\alpha'}F_0^{\alpha'}{}_{\nu'}], \quad (8.114)$$

$$[\chi_{2,1;(\mu\nu)}] = -2[E_{1,(\mu';\nu)}] - 2[E_{0,\alpha';(\mu}F_0^{\alpha'}{}_{\nu)}], \quad (8.115)$$

$$[\chi_{3,1;(\mu\nu)}] = -2[\tilde{E}_{1,(\mu';\nu)}] - 2[F_{0,(\mu'|\alpha'}|\tilde{E}_{0;\nu)}^{\alpha'}]. \quad (8.116)$$

Next, we compute

$$[E_{1,\mu';\nu}] = -\frac{1}{N}\text{tr}[\nabla_\nu\nabla_{\mu'}a_1] - \frac{2}{N}\text{tr}[\mathcal{P}_{;\mu'\nu}]Q, \quad (8.117)$$

$$[\tilde{E}_{1,\mu';\nu}] = \frac{1}{N}\text{tr}[\nabla_\nu\nabla_{\mu'}a_1] + \frac{2}{N}\text{tr}[\mathcal{P}_{;\mu\nu}]Q. \quad (8.118)$$

Therefore,

$$[\chi_{1,2;(\mu\nu)}] = \frac{4}{N} \text{tr} [\nabla_{(\mu'} \nabla_{\nu)} a_1] + \frac{4}{9} R_{\mu\alpha} R_{\nu}^{\alpha}, \quad (8.119)$$

$$[\chi_{2,1;(\mu\nu)}] = \frac{2}{N} \text{tr} [\nabla_{(\mu'} \nabla_{\nu)} a_1] + \frac{1}{18} R_{\mu\alpha} R_{\nu}^{\alpha}, \quad (8.120)$$

$$[\chi_{3,1;(\mu\nu)}] = -\frac{2}{N} \text{tr} [\nabla_{(\mu} \nabla_{\nu)} a_1] + \frac{1}{18} R_{\mu\alpha} R_{\nu}^{\alpha}, \quad (8.121)$$

and

$$[\Phi_{1;(\mu\nu)}] = -\frac{2}{N} \text{tr} [\nabla_{(\mu} \nabla_{\nu)} a_1] - \frac{1}{9} R_{\mu\alpha} R_{\nu}^{\alpha}. \quad (8.122)$$

Therefore,

$$[\Phi_1{}^{;\mu}{}_{\mu}] = -\frac{2}{N} \text{tr} [\nabla^{\mu} \nabla_{\mu} a_1] - \frac{1}{9} R_{\mu\nu} R^{\mu\nu}. \quad (8.123)$$

By using the known formula for the coefficient $[\nabla^{\mu} \nabla_{\mu} a_1]$ (see Appendix) we get

$$[\Phi_1{}^{;\mu}{}_{\mu}] = -\frac{2}{3N} \text{tr} \nabla^{\mu} \nabla_{\mu} Q + \frac{1}{3N} \text{tr} \mathcal{R}_{\alpha\mu} \mathcal{R}^{\alpha\mu} + \frac{2}{15} \nabla^{\mu} \nabla_{\mu} R \quad (8.124)$$

$$-\frac{2}{15} R_{\mu\nu} R^{\mu\nu} + \frac{1}{45} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}. \quad (8.125)$$

We still need $[J_1{}^{;\mu}{}_{\mu}]$ and $[\Lambda_1{}^{;\mu}{}_{\mu}]$.

$$J_1 = -2F_0{}^{\mu'}{}_{\mu'}. \quad (8.126)$$

We have

$$F_0{}^{\alpha'}{}_{\beta'} = \frac{1}{N} \gamma^{\alpha'\delta} \text{tr} \left(\zeta_{;\delta\beta'} + \zeta_{;\delta} \zeta_{;\beta'} + \zeta_{;\delta} \mathcal{P}^{-1} \mathcal{P}_{;\beta'} + \zeta_{;\beta'} \mathcal{P}^{-1} \mathcal{P}_{;\delta} + \mathcal{P}^{-1} \mathcal{P}_{;\delta\beta'} \right), \quad (8.127)$$

then

$$\begin{aligned} [F_0{}^{\alpha'}{}_{\beta';\mu\nu}] &= -\frac{1}{6} [\gamma^{\alpha'\delta}{}_{;\mu\nu}] R_{\delta\beta} - \frac{1}{N} g^{\alpha\delta} \text{tr} \left[\zeta_{;\delta\beta'}{}_{\mu\nu} + 2\zeta_{;\delta(\mu} \zeta_{;\beta')\nu} + 2\zeta_{;\delta(\mu} \mathcal{P}_{;\beta')\nu} \right. \\ &\quad \left. + 2\zeta_{;\beta'(\mu} \mathcal{P}_{;\delta)\nu} - \mathcal{P}_{;\mu\nu} \mathcal{P}_{;\delta\beta'} + \mathcal{P}_{;\delta\beta'}{}_{\mu\nu} \right], \end{aligned} \quad (8.128)$$

thus

$$[F_0{}^{\alpha'}{}_{\beta';\mu\nu}] = -\frac{1}{6} [\gamma^{\alpha'\delta}{}_{;\mu\nu}] R_{\delta\beta} - [\zeta^{\alpha}{}_{\beta'\mu\nu}] - \frac{2}{36} R^{\alpha}{}_{(\mu} R_{\beta)\nu} + \frac{1}{4N} \text{tr} \mathcal{R}_{\mu\nu} \mathcal{R}^{\alpha}{}_{\beta} - \frac{1}{N} \text{tr} [\mathcal{P}^{\alpha}{}_{\beta'\mu\nu}]. \quad (8.129)$$

Therefore we have

$$[F_0^{\mu'}{}_{\mu'}{}^{;\nu}{}_{\nu}] = -[\zeta^{\mu}{}_{\mu'}{}^{;\nu}{}_{\nu}] - \frac{1}{N} \text{tr} [\mathcal{P}^{\mu}{}_{\mu'}{}^{;\nu}{}_{\nu}]. \quad (8.130)$$

Simplifying, we find

$$[F_0^{\mu'}{}_{\mu'}{}^{;\nu}{}_{\nu}] = \frac{1}{30} \nabla_{\mu} \nabla^{\mu} R + \frac{1}{45} R_{\mu\nu} R^{\mu\nu} + \frac{1}{30} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - \frac{1}{2N} \text{tr} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu}. \quad (8.131)$$

Then we have

$$[J_1^{\mu}{}_{\mu}] = -2[F_0^{\mu'}{}_{\mu'}{}^{;\nu}{}_{\nu}] = -\frac{2}{30} \nabla_{\mu} \nabla^{\mu} R - \frac{2}{45} R_{\mu\nu} R^{\mu\nu} - \frac{2}{30} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} + \frac{1}{N} \text{tr} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu}. \quad (8.132)$$

Next, calculating $[\Lambda_{1;\mu\nu}]$, we have

$$\Lambda_{1;\mu\nu} = -\text{tr} (a_{1;\mu\nu} + a_{1;\mu\nu}^*), \quad (8.133)$$

thus

$$[\Lambda_1^{\mu}{}_{\mu}] = -2\text{tr} [\nabla_{\mu} \nabla^{\mu} a_1]. \quad (8.134)$$

Finally, by collecting all the terms, we get

$$\begin{aligned} h_{1,1} = & \frac{n^2 - n - 2}{6n} N^{n-1} \text{tr} \nabla^{\mu} \nabla_{\mu} Q + \frac{2 + 3n - 2n^2}{60n} N^n \nabla^{\mu} \nabla_{\mu} R \\ & + \frac{8 + 4n - n^2}{12n} N^{n-1} \text{tr} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} - n N^{n-1} \text{tr} Q^2 - n(n-1) N^{n-2} (\text{tr} Q)^2 \\ & - \frac{n(n+3)}{3} N^{n-1} R \text{tr} Q + \frac{3n^2 + 2n - 5}{36} N^n R^2 + \frac{n^2 + n - 16}{180n} N^n R_{\mu\nu} R^{\mu\nu} \\ & + \frac{-n^2 + 4n - 4}{180n} N^n R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}. \end{aligned} \quad (8.135)$$

Next, we compute $h_{0,2}$. We have

$$\begin{aligned} h_{0,\mu\nu\alpha\beta} = & \left[N^n \Phi_0 (2n \zeta_{(\mu\nu\alpha\beta)} + 24n^2 \zeta_{(\mu\nu} \zeta_{\alpha\beta)}) \right. \\ & + 6N^n \zeta_{(\mu\nu} \Phi_{0;\alpha\beta)} - 6N^{n-1} \zeta_{(\mu\nu} g_{\alpha\beta)} (N J_1 + n \Lambda_1) \\ & \left. + N^n \Phi_{0;(\mu\nu\alpha\beta)} - 6N^n g_{(\mu\nu} J_{1;\alpha\beta)} - 6n N^{n-1} g_{(\mu\nu} \Lambda_{1;\alpha\beta)} \right]. \end{aligned} \quad (8.136)$$

Therefore,

$$\begin{aligned}
h_{0,2} = & \frac{1}{8n^2} \left(N^n [\Phi_0] \left(2n[\zeta_{(4)}] + \frac{24n^2}{108} (R^2 + 2R_{\mu\nu}R^{\alpha\beta}) \right) \right. \\
& + N^n \frac{1}{3} (R[\Phi_{0;\mu}{}^\mu] + 2R^{\mu\nu}[\Phi_{0;(\mu\nu)}]) - N^{n-1} \frac{n+2}{3} R(N[J_1] + n[\Lambda_1]) \\
& + N^n g^{\mu\nu} g^{\alpha\beta} [\Phi_{;(\mu\nu\alpha\beta)}] - 2N^n (n+2) [J_1;{}^\mu{}_\mu] \\
& \left. - 2N^{n-1} n(n+2) [\Lambda_1;{}^\mu{}_\mu] \right). \tag{8.137}
\end{aligned}$$

We already know $[\zeta_{(4)}]$, $[\Phi_0]$, $[\Phi_{0;(\mu\nu)}]$, $[J_1]$, $[\Lambda_1]$, $[J_1;{}^\mu{}_\mu]$, and $[\Lambda_1;{}^\mu{}_\mu]$. We calculate $[\Phi_{0;(\alpha\beta\mu\nu)}]$. We find

$$\Phi_{0;(\alpha\beta\mu\nu)} = -\frac{1}{2} \chi_{1,1;(\alpha\beta\mu\nu)} + \chi_{2,0;(\alpha\beta\mu\nu)} + \chi_{3,0;(\alpha\beta\mu\nu)} + 6\chi_{2,0;(\alpha\beta)} \chi_{3,0;[\mu\nu]} - 6\chi_{1,0;(\alpha\beta)} \chi_{4,0;[\mu\nu]}. \tag{8.138}$$

We have

$$\chi_{1,1;(\alpha\beta\mu\nu)} = 24\sigma_{\delta'(\alpha} F_0{}^{\delta'\gamma'}{}_{;\beta\mu} \sigma_{|\gamma'| \nu)}, \tag{8.139}$$

then

$$g^{\alpha\beta} g^{\mu\nu} [\chi_{1,1;(\alpha\beta\mu\nu)}] = 8 \left([F_0{}^{\mu'}{}_{\mu'}{}^{;\nu}{}_\nu] + 2[F_0{}^{\mu'v'}{}_{;\mu\nu}] \right). \tag{8.140}$$

We know $[F_0{}^{\mu'}{}_{\mu'}{}^{;\nu}{}_\nu]$. Next,

$$[F_0{}^{\mu'v'}{}_{;\mu\nu}] = -\frac{1}{6} [\gamma^{\mu'\alpha}{}_{;\mu\nu}] R_\alpha{}^\nu - [\zeta^{\mu\nu'}{}_{\mu\nu}] + \frac{1}{36} (R^2 + R_{\mu\nu}R^{\mu\nu}) - \frac{1}{4N} \text{tr} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} - \frac{1}{N} [\mathcal{P}^{\mu\nu'}{}_{\mu\nu}]. \tag{8.141}$$

Simplifying, we obtain

$$[F_0{}^{\mu'v'}{}_{;\mu\nu}] = \frac{1}{36} R^2 + \frac{1}{30} \nabla_\mu \nabla^\mu R - \frac{7}{60} R_{\mu\nu} R^{\mu\nu} + \frac{1}{30} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} + \frac{1}{4N} \text{tr} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu}. \tag{8.142}$$

Thus

$$g^{\alpha\beta} g^{\mu\nu} [\chi_{1,1;(\alpha\beta\mu\nu)}] = \frac{4}{9} R^2 + \frac{4}{5} \nabla_\mu \nabla^\mu R - \frac{19}{90} R_{\mu\nu} R^{\mu\nu} + \frac{1}{10} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}. \tag{8.143}$$

We have

$$[\chi_{2,0;(\alpha\beta\mu\nu)}] = [4E_{0,\delta';(\alpha\beta\mu} \sigma^{\delta'}{}_\nu)] \tag{8.144}$$

So

$$[E_{0,\delta';(\alpha\beta\mu)}] = \frac{1}{N} \text{tr} \left[\zeta_{;\delta'(\alpha\beta\mu)} + \mathcal{P}_{;\delta'(\alpha\beta\mu)} + 2\mathcal{P}_{;(\alpha\beta}^{-1} \mathcal{P}_{;|\delta'\mu)} \right] \tag{8.145}$$

Thus we find

$$[\chi_{2,0;(\alpha\beta\mu\nu)}] = 4[\zeta_{;(\alpha'\beta\mu\nu)}] + \frac{4}{N}\text{tr}[\mathcal{P}_{;(\alpha'\beta\mu\nu)}] - \frac{12}{N}\text{tr}[\mathcal{P}_{;\alpha\beta}\mathcal{P}_{;\mu'\nu}]. \quad (8.146)$$

Then

$$g^{\alpha\beta}g^{\mu\nu}[\chi_{2,0;(\alpha\beta\mu\nu)}] = -\frac{2}{15}\nabla_\mu\nabla^\mu R - \frac{2}{135}R_{\mu\nu}R^{\mu\nu} - \frac{2}{15}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - \frac{2}{N}\text{tr}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}. \quad (8.147)$$

Next, we have

$$[\chi_{3,0;(\alpha\beta\mu\nu)}] = [4\sigma_{\delta'(\alpha}\tilde{E}_{0;\beta\mu\nu)}^{\delta'}]. \quad (8.148)$$

We compute

$$\begin{aligned} [\tilde{E}_{0;(\beta\mu\nu)}^{\alpha'}] &= [\gamma^{\alpha\delta}]\left([\zeta_{;\delta(\beta\mu\nu)}] + \frac{1}{N}\text{tr}[\mathcal{P}_{;\delta'(\beta\mu\nu)}] + \frac{3}{N}\text{tr}[\mathcal{P}_{;(\beta\mu}^{-1}\mathcal{P}_{;|\delta|\nu)}]\right) \\ &\quad + \frac{1}{N}[\gamma^{\alpha'\delta}_{;(\beta\mu)}]\text{tr}[\zeta_{;|\delta|\nu}] + \frac{1}{N}[\gamma^{\alpha'\delta}_{;(\beta\mu)}]\text{tr}[\mathcal{P}_{;|\delta|\nu}]. \end{aligned} \quad (8.149)$$

Then

$$[\chi_{3,0;(\alpha\beta\mu\nu)}] = 4[\zeta_{;(\alpha\beta\mu\nu)}] - \frac{12}{N}\text{tr}[\mathcal{P}_{;(\alpha\beta}\mathcal{P}_{;\mu\nu)}] - 4[\gamma^{\delta'\gamma}_{;(\alpha\beta)}][\zeta_{;|\gamma|\mu}g_{|\delta|\nu}]. \quad (8.150)$$

Then we obtain

$$g^{\alpha\beta}g^{\mu\nu}[\chi_{3,0;(\alpha\beta\mu\nu)}] = \frac{4}{5}\nabla_\mu\nabla^\mu R + \frac{4}{45}R_{\mu\nu}R^{\mu\nu} + \frac{2}{15}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - \frac{2}{N}\text{tr}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}. \quad (8.151)$$

Lastly we need

$$[\chi_{4,0;\mu\nu}] = [E_{0,\alpha'(\mu}\tilde{E}_{0,\nu)}^{\alpha'}]. \quad (8.152)$$

We find

$$[E_{0,\alpha'\mu}] = -\frac{1}{6}R_{\alpha\mu} \quad (8.153)$$

and

$$[\tilde{E}_{0,\nu}^{\alpha'}] = -\frac{1}{6}R^\alpha{}_\nu, \quad (8.154)$$

thus

$$[\chi_{4,0;(\mu\nu)}] = \frac{1}{36}R_{\alpha(\mu}R^\alpha{}_{\nu)}. \quad (8.155)$$

Thus we have

$$g^{\alpha\beta}g^{\mu\nu}[\chi_{2,0;(\alpha\beta}\chi_{3,0;\mu\nu)}] = \frac{1}{27}(R^2 + 2R_{\mu\nu}R^{\mu\nu}) \quad (8.156)$$

and

$$g^{\alpha\beta} g^{\mu\nu} [\chi_{1,0;(\alpha\beta\chi_{4,0;\mu\nu})}] = \frac{n+2}{54} R_{\mu\nu} R^{\mu\nu}. \quad (8.157)$$

Putting this all together, we obtain

$$g^{\alpha\beta} g^{\mu\nu} [\Phi_{0;(\alpha\beta\mu\nu)}] = \frac{4}{15} \nabla_\mu \nabla^\mu R - \frac{60n-217}{540} R_{\mu\nu} R^{\mu\nu} - \frac{1}{20} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - \frac{4}{N} \text{tr} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu}, \quad (8.158)$$

which gives

$$\begin{aligned} h_{0,2} = & \frac{n^2 - n + 3}{72n^2} N^n R^2 + \frac{n^2 + 8}{120n^2} N^n \nabla_\mu \nabla^\mu R + \frac{264n^2 + 60n + 433}{4320n^2} N^n R_{\mu\nu} R^{\mu\nu} \\ & - \frac{8n^2 - 20n - 39}{1440n^2} N^n R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - \frac{2n^2 + 10n + 24}{24n^2} N^{n-1} \text{tr} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} \\ & + \frac{n+2}{12n} N^{n-1} R \text{tr} Q + \frac{n+2}{6n} N^{n-1} \nabla_\mu \nabla^\mu \text{tr} Q. \end{aligned} \quad (8.159)$$

Thus we finally have

$$\begin{aligned} b_2 = & N^n \left\{ \frac{20n^4 - 8n^3 - 11n^2 - 6n + 6}{144n^2} R^2 + \frac{4n^3 + 11n^2 + n - 4}{120n^2} \nabla_\mu \nabla^\mu R \right. \\ & + \frac{-24n^3 + 84n^2 - 576n + 385}{4320n^2} R_{\mu\nu} R^{\mu\nu} + \frac{8n^3 - 8n^2 - 18n + 15}{1440n^2} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \Big\} \\ & + \frac{n^3 - n^2 + 3n - 12}{12n^2} N^{n-1} \text{tr} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + \frac{-12n^3 - 4n^2 + n + 2}{12n} N^{n-1} R \text{tr} Q \\ & - \frac{n+2}{6} N^{n-1} \text{tr} \nabla^\mu \nabla_\mu Q + n N^{n-1} \text{tr} Q^2 + n(n-1) N^{n-2} (\text{tr} Q)^2. \end{aligned} \quad (8.160)$$

A Appendix

A.1 Derivatives of Sygne Function

We list the coincidence limits of the symmetrized derivatives of the two-point functions introduced in Sec. 5. A very efficient algorithm for computing such coincidence limits of the derivatives of two-point functions is developed in [1, 2]. All the formulas below are computed there.

The coincidence limits of mixed derivatives of a two-point function $f = f(x, x')$ can be computed by using the equation

$$[\nabla_{\mu'} f] = \nabla_\mu [f] - [\nabla_\mu f]. \quad (A.1)$$

First, the coincidence limits of the Synge function and its first derivatives vanish,

$$[\sigma] = [\sigma^\mu] = [\sigma^{\mu'}] = 0. \quad (\text{A.2})$$

The coincidence limits of the second derivatives are

$$[\sigma_{\mu\nu}] = -[\sigma_{\mu\nu'}] = g_{\mu\nu}, \quad (\text{A.3})$$

and the coincidence limits of higher-order symmetrized derivatives of the vectors σ^μ and $\sigma^{\mu'}$ vanish,

$$[\sigma^\alpha_{(\mu_1 \dots \mu_k)}] = [\sigma^{\alpha'}_{(\mu_1 \dots \mu_k)}] = 0, \quad k \geq 2. \quad (\text{A.4})$$

All other coincidence limits can be obtained from these by commuting covariant derivatives.

The coincidence limits of the low-order symmetrized derivatives have the form [2, 5]

$$[\sigma^\alpha_{\beta\mu}] = [\sigma^{\alpha'}_{\beta\mu}] = 0, \quad (\text{A.5})$$

$$[\sigma^\alpha_{\beta\mu_1\mu_2}] = -\frac{2}{3}R^\alpha_{(\mu_1|\beta|\mu_2)}, \quad (\text{A.6})$$

$$[\sigma^{\alpha'}_{\beta(\mu_1\mu_2)}] = -\frac{1}{3}R^\alpha_{(\mu_1|\beta|\mu_2)}, \quad (\text{A.7})$$

$$[\sigma^{\alpha'}_{\beta\mu_1\mu_2}] = -\frac{2}{3}R^\alpha_{(\beta\mu_1)\mu_2}, \quad (\text{A.8})$$

$$[\sigma^{\alpha'}_{\beta'\mu_1\mu_2}] = -\frac{2}{3}R^\alpha_{(\mu_1|\beta|\mu_2)}, \quad (\text{A.9})$$

$$[\sigma^\alpha_{\beta(\mu_1\mu_2\mu_3)}] = -\frac{3}{2}\nabla_{(\mu_1}R^\alpha_{\mu_2|\beta|\mu_3)}, \quad (\text{A.10})$$

$$[\sigma^{\alpha'}_{\beta(\mu_1\mu_2\mu_3)}] = -\frac{1}{2}\nabla_{(\mu_1}R^\alpha_{\mu_2|\beta|\mu_3)}, \quad (\text{A.11})$$

$$[\sigma^\alpha_{\beta(\mu_1\mu_2\mu_3\mu_4)}] = -\frac{12}{5}\nabla_{(\mu_1}\nabla_{\mu_2}R^\alpha_{\mu_3|\beta|\mu_4)} - \frac{8}{15}R^\alpha_{(\mu_1|\gamma|\mu_2}R^\gamma_{\mu_3|\beta|\mu_4)}, \quad (\text{A.12})$$

$$[\sigma^{\alpha'}_{\beta(\mu_1\mu_2\mu_3\mu_4)}] = -\frac{3}{5}\nabla_{(\mu_1}\nabla_{\mu_2}R^\alpha_{\mu_3|\beta|\mu_4)} - \frac{7}{15}R^\alpha_{(\mu_1|\gamma|\mu_2}R^\gamma_{\mu_3|\beta|\mu_4)}, \quad (\text{A.13})$$

$$[\gamma^\alpha_{\beta'}] = -\delta^\alpha_{\beta'}, \quad (\text{A.14})$$

$$[\gamma^\alpha_{\beta';\mu}] = 0, \quad (\text{A.15})$$

$$[\gamma^\alpha_{\beta';(\mu_1\mu_2)}] = \frac{1}{3}R^\alpha_{(\mu_1|\beta|\mu_2)}, \quad (\text{A.16})$$

$$[\gamma^\alpha_{\beta';\mu_1\mu_2}] = \frac{2}{3}R^\alpha_{(\beta\mu_1)\mu_2}, \quad (\text{A.17})$$

$$[\gamma^\alpha_{\beta';(\mu_1\mu_2\mu_3)}] = \frac{1}{2}\nabla_{(\mu_1}R^\alpha_{\mu_2|\beta|\mu_3)}, \quad (\text{A.18})$$

$$[\gamma^\alpha_{\beta';(\mu_1\mu_2\mu_3\mu_4)}] = \frac{3}{5}\nabla_{(\mu_1}\nabla_{\mu_2}R^\alpha_{\mu_3|\beta|\mu_4)} - \frac{1}{5}R^\alpha_{(\mu_1|\gamma|\mu_2}R^\gamma_{\mu_3|\beta|\mu_4)}. \quad (\text{A.19})$$

The coincidence limit of the function $\zeta = \frac{1}{2} \log \Delta$ and its derivative vanish

$$[\zeta] = [\zeta_{;\mu}] = 0. \quad (\text{A.20})$$

By using the eq. (5.6) we find for the higher-order derivatives

$$[\zeta_{;(\mu_1\dots\mu_k)}] = -\frac{1}{2}[\sigma^\alpha_{\alpha(\mu_1\dots\mu_k)}], \quad k \geq 1. \quad (\text{A.21})$$

All higher-order derivatives can be computed by commuting derivatives.

We will need the following derivatives

$$[\zeta_{;\mu_1\mu_2}] = \frac{1}{6}R_{\mu_1\mu_2}, \quad (\text{A.22})$$

$$[\zeta_{;\mu\nu}] = -\frac{1}{6}R_{\mu\nu}, \quad (\text{A.23})$$

$$[\zeta_{;(\mu_1\mu_2\mu_3)}] = \frac{1}{4}\nabla_{(\mu_1}R_{\mu_2\mu_3)}, \quad (\text{A.24})$$

$$[\zeta_{;(\mu_1\mu_2\mu_3\mu_4)}] = \frac{3}{10}\nabla_{(\mu_1}\nabla_{\mu_2}R_{\mu_3\mu_4)} + \frac{1}{15}R_{\alpha(\mu_1|\gamma|\mu_2}R^\gamma_{\mu_3\mu_4)}{}^\alpha. \quad (\text{A.25})$$

We compute some contractions

$$g^{\mu_1\mu_2}g^{\mu_3\mu_4}[\zeta_{;(\mu_1\mu_2\mu_3\mu_4)}] = \frac{1}{5}\nabla_\mu\nabla^\mu R + \frac{1}{45}R_{\mu\nu}R^{\mu\nu} + \frac{1}{30}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, \quad (\text{A.26})$$

$$[\zeta^{\mu'}_{\mu}{}^{\nu}_{\nu}] = -\frac{1}{30}\nabla_\mu\nabla^\mu R - \frac{1}{45}R_{\mu\nu}R^{\mu\nu} - \frac{1}{30}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, \quad (\text{A.27})$$

$$[\zeta^{\mu\nu'}_{\mu\nu}] = -\frac{1}{30}\nabla_\mu\nabla^\mu R + \frac{4}{45}R_{\mu\nu}R^{\mu\nu} - \frac{1}{30}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, \quad (\text{A.28})$$

For the operator of parallel transport, the coincidence limit is equal to the identity matrix,

$$[\mathcal{P}] = \mathbb{I} \quad (\text{A.29})$$

and the coincidence limits of higher-order symmetrized derivatives vanish,

$$[\mathcal{P}_{;(\alpha\mu_1\dots\mu_k)}] = [\mathcal{P}_{;(\alpha'\mu_1\dots\mu_k)}] = 0, \quad k \geq 0. \quad (\text{A.30})$$

In particular, it is easy to get

$$[\mathcal{P}_{;\mu}] = [\mathcal{P}_{;\mu'}] = 0. \quad (\text{A.31})$$

All other coincidence limits can be obtained from these by commuting derivatives.

It is convenient to introduce the following vector [2, 1]

$$\mathcal{A}_{\mu'} = \mathcal{P}^{-1} \gamma_{\mu'}^{\nu} \nabla_{\nu} \mathcal{P}, \quad (\text{A.32})$$

so that the first derivative of the parallel transport operator has the form

$$\mathcal{P}_{;\nu} = \mathcal{P} \sigma^{\mu'}_{\nu} \mathcal{A}_{\mu'}. \quad (\text{A.33})$$

It is easy to see that

$$[\mathcal{A}_{\mu'}] = 0. \quad (\text{A.34})$$

Further, some low-order symmetrized derivatives of the vector $\mathcal{A}_{\mu'}$ are

$$[\mathcal{A}^{\nu'}_{;\mu_1}] = \frac{1}{2} \mathcal{R}^{\nu}_{\mu_1}, \quad (\text{A.35})$$

$$[\mathcal{A}^{\nu'}_{;(\mu_1\mu_2)}] = \frac{2}{3} \nabla_{(\mu_1} \mathcal{R}^{\nu}_{\mu_2)}, \quad (\text{A.36})$$

$$[\mathcal{A}^{\nu'}_{;(\mu_1\mu_2\mu_3)}] = \frac{3}{4} \nabla_{(\mu_1} \nabla_{\mu_2} \mathcal{R}^{\nu}_{\mu_3)} - \frac{1}{4} R^{\nu}_{(\mu_1|\alpha|\mu_2} \mathcal{R}^{\alpha}_{\mu_3)}. \quad (\text{A.37})$$

By using the covariant Taylor expansion of this vector one can compute all coincidence limits of the parallel transport operator we need. Some low-order derivatives have the form

$$[\mathcal{P}_{;\nu\mu}] = -[\mathcal{P}_{;\nu\mu'}] = -\frac{1}{2} \mathcal{R}_{\nu\mu}, \quad (\text{A.38})$$

$$[\mathcal{P}^{;\nu}_{\mu_1\mu_2}] = -\frac{2}{3} \nabla_{(\mu_1} \mathcal{R}^{\nu}_{\mu_2)}, \quad (\text{A.39})$$

$$[\mathcal{P}^{;\nu'}_{\mu_1\mu_2}] = -\frac{1}{3} \nabla_{(\mu_1} \mathcal{R}^{\nu}_{\mu_2)}, \quad (\text{A.40})$$

$$[\mathcal{P}^{;\nu}_{(\mu_1\mu_2\mu_3)}] = -\frac{3}{4} \nabla_{(\mu_1} \nabla_{\mu_2} \mathcal{R}^{\nu}_{\mu_3)} - \frac{1}{4} R^{\nu}_{(\mu_1|\alpha|\mu_2} \mathcal{R}^{\alpha}_{\mu_3)}, \quad (\text{A.41})$$

$$[\mathcal{P}^{;\nu'}_{(\mu_1\mu_2\mu_3)}] = \frac{1}{12} \nabla_{(\mu_1} \nabla_{\mu_2} \mathcal{R}^{\nu}_{\mu_3)} + \frac{1}{4} R^{\nu}_{(\mu_1|\alpha|\mu_2} \mathcal{R}^{\alpha}_{\mu_3)}. \quad (\text{A.42})$$

We will need the following contractions

$$\mathrm{tr} [\mathcal{P}^{i\mu}{}_{\mu'}{}^\nu{}_\nu] = \frac{1}{2} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu}, \quad (\text{A.43})$$

$$\mathrm{tr} [\mathcal{P}^{i\mu\nu'}{}_{\mu\nu}] = -\frac{1}{2} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu}, \quad (\text{A.44})$$

$$\mathrm{tr} [\mathcal{P}^{i\mu\nu'}{}_{\nu\mu}] = 0. \quad (\text{A.45})$$

A.2 Derivatives of the Heat Kernel Coefficients

We list below the coincidence limits of the heat kernel coefficients and their derivatives (see [2, 5]; notice some different sign conventions)

$$[a_0] = \mathbb{I}, \quad (\text{A.46})$$

$$[a_1] = Q - \frac{1}{6} R, \quad (\text{A.47})$$

$$[\nabla_\mu a_1] = \frac{1}{2} \nabla_\mu Q - \frac{1}{12} \nabla_\mu R + \frac{1}{6} \nabla_\nu \mathcal{R}^\nu{}_\mu, \quad (\text{A.48})$$

$$\begin{aligned} [\nabla_{(\mu} \nabla_{\nu)} a_1] &= \frac{1}{3} \nabla_{(\mu} \nabla_{\nu)} Q - \frac{1}{6} \mathcal{R}_{\alpha(\mu} \mathcal{R}^{\alpha}{}_{\nu)} + \frac{1}{6} \nabla_{(\mu} \nabla_{|\alpha|} \mathcal{R}^{\alpha}{}_{\nu)} - \frac{1}{20} \nabla_\mu \nabla_\nu R \\ &\quad - \frac{1}{60} \nabla_\alpha \nabla^\alpha R_{\mu\nu} + \frac{1}{45} R_{\mu\alpha} R^\alpha{}_\nu - \frac{1}{90} R_{\mu\alpha\beta\gamma} R^\alpha{}_\nu{}^{\beta\gamma} - \frac{1}{90} R_{\alpha\beta} R^\alpha{}_\mu{}^\beta{}_\nu, \end{aligned} \quad (\text{A.49})$$

$$\begin{aligned} [\nabla^\mu \nabla_\mu a_1] &= \frac{1}{3} \nabla^\mu \nabla_\mu Q - \frac{1}{6} \mathcal{R}_{\alpha\mu} \mathcal{R}^{\alpha\mu} - \frac{1}{15} \nabla^\mu \nabla_\mu R \\ &\quad + \frac{1}{90} R_{\mu\nu} R^{\mu\nu} - \frac{1}{90} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}, \end{aligned} \quad (\text{A.50})$$

$$\begin{aligned} [a_2] &= \left(Q - \frac{1}{6} R \right)^2 - \frac{1}{3} \nabla^\mu \nabla_\mu Q + \frac{1}{6} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + \frac{1}{15} \nabla^\mu \nabla_\mu R \\ &\quad - \frac{1}{90} R_{\mu\nu} R^{\mu\nu} + \frac{1}{90} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}. \end{aligned} \quad (\text{A.51})$$

A.3 Gaussian Integrals

Let $A = (A_{ij})$ be a real symmetric positive matrix. Then for any vector $B = (B_i)$ there holds

$$\begin{aligned} \int_{\mathbb{R}^n} d\xi \exp(-\langle \xi, A\xi \rangle + \langle B, \xi \rangle) \\ = \pi^{n/2} (\det A)^{-1/2} \exp\left(\frac{1}{4} \langle B, A^{-1} B \rangle\right), \end{aligned} \quad (\text{A.52})$$

where $A^{-1} = (A^{ij})$ is the inverse of the matrix A . By expanding both sides of eq. (A.52) in Taylor series in B_i we obtain

$$\int_{\mathbb{R}^n} d\xi \exp(-\langle \xi, A\xi \rangle) \xi^{i_1} \dots \xi^{i_{2k+1}} = 0, \quad (\text{A.53})$$

$$\begin{aligned} \int_{\mathbb{R}^n} d\xi \exp(-\langle \xi, A\xi \rangle) \xi^{i_1} \dots \xi^{i_{2k}} \\ = \pi^{n/2} (\det A)^{-1/2} \frac{(2k)!}{2^{2k} k!} A^{(i_1 i_2 \dots i_{2k})}, \end{aligned} \quad (\text{A.54})$$

where the parenthesis denote complete symmetrization over all indices included.

We introduce the Gaussian average

$$\langle f \rangle = \pi^{-n/2} (\det A)^{1/2} \int_{\mathbb{R}^n} d\xi \exp(-\langle \xi, A\xi \rangle) f(\xi). \quad (\text{A.55})$$

Then, the above formulas can be written in the form

$$\langle \xi^{i_1} \dots \xi^{i_{2k+1}} \rangle = 0, \quad (\text{A.56})$$

$$\langle \xi^{i_1} \dots \xi^{i_{2k}} \rangle = \frac{(2k)!}{2^{2k} k!} A^{(i_1 i_2 \dots i_{2k})}. \quad (\text{A.57})$$

A.4 Leibnitz Rule

In this paper we extensively use the Leibnitz rule for the symmetrized derivative of the product

$$\nabla_{(\mu_1} \dots \nabla_{\mu_n)} (fg) = \sum_{k=0}^n \binom{n}{k} (\nabla_{(\mu_1} \dots \nabla_{\mu_k} f) (\nabla_{\mu_{k+1}} \dots \nabla_{\mu_n)} g). \quad (\text{A.58})$$

In particular,

$$\begin{aligned} \nabla_{(\mu_1} \cdots \nabla_{\mu_4)}(fg) &= f_{;(\mu_1\mu_2\mu_3\mu_4)}g + 4f_{;(\mu_1\mu_2\mu_3}g_{;\mu_4)} + 6f_{;(\mu_1\mu_2}g_{;\mu_3\mu_4)} \\ &\quad + 4f_{;(\mu_1}g_{;\mu_2\mu_3\mu_4)} + fg_{;(\mu_1\mu_2\mu_3\mu_4)}. \end{aligned} \quad (\text{A.59})$$

Let $f = e^h$. Then

$$f_{;\mu} = fh_{;\mu}, \quad (\text{A.60})$$

$$f_{;(\mu_1\mu_2)} = f(h_{;(\mu_1\mu_2)} + h_{;(\mu_1}h_{;\mu_2)}), \quad (\text{A.61})$$

$$f_{;(\mu_1\mu_2\mu_3)} = f(h_{;(\mu_1\mu_2\mu_3)} + 3h_{;(\mu_1\mu_2}h_{;\mu_3)} + h_{;(\mu_1}h_{;\mu_2}h_{;\mu_3)}), \quad (\text{A.62})$$

$$\begin{aligned} f_{;(\mu_1\mu_2\mu_3\mu_4)} &= f\left(h_{;(\mu_1\mu_2\mu_3\mu_4)} + 3h_{;(\mu_1\mu_2}h_{;\mu_3\mu_4)} + 4h_{;(\mu_1}h_{;\mu_2\mu_3\mu_4)}\right. \\ &\quad \left.+ 6h_{;(\mu_1}h_{;\mu_2}h_{;\mu_3\mu_4)} + h_{;(\mu_1}h_{;\mu_2}h_{;\mu_3}h_{;\mu_4)}\right). \end{aligned} \quad (\text{A.63})$$

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